The Mathematics Teacher

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The "make 'em run" theory
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New fields and organizations in the mathematical sciences[1]

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New uses of mathematics generate new mathematical ideas for study. This new age of mathematics is reflected in the growth of mathematical organizations.

MATHEMATICS IS ENTERING the greatest era this venerable discipline has ever known. A quarter of a century ago the stereotyped conception of mathematical education was that its basic purpose at the high school level was to prepare for the study of mathematics in college, and that the study of mathematics in college could lead in only one of two directions: First, as a tooling-up discipline for engineers and physicists. Second, as preparation for a career in the teaching of mathematics at the secondary school, college, or university level, with research and scholarly writing as an overtime activity for the more able and energetic. It is true that persons with Ph.D. training in mathematics could occasionally be found doing work of a mathematical nature in government and industry at that time. And not quite all college mathematics majors became mathematics teachers. A few became actuaries or computing machine operators or simply used their mathematical training as background for preparation in law or some other profession. But times have changed. In recent years, especially since World War II, there has been an unprecedented demand for persons with mathematical talent and training. This demand has come from many different quarters. There has been a great expansion of interest and research in pure mathematics. There are trends toward greater emphasis on mathematical training in the physical sciences and engineering. There is a new movement under way for wider use of mathematical methods in dealing with some of the problems of the social and biological sciences. Entirely new fields are opening up in industry, business, and government for specialists in the various mathematical sciences.

This mathematical boom is accompanied by all sorts of new developments and problems. The growing requirements for mathematical training are placing a mounting burden on the mathematics teachers now in service. The new demands for mathematical talent are raising questions of more effective recruitment not only for filling new positions in industry, business, and government, but for staffing the teaching profession itself. The rise of new fields of mathematical science and the newly discovered interests of the social and biological sciences in the possibilities of mathematical methods in dealing with their problems have pointed up the shortcomings of the mathematics curriculum as we have known it. One of the most significant features of the new mathematical expansion in this country has been the birth and growth of national societies in

[1] Talk given at a dinner meeting of the National Science Foundation Summer Institute for High School Mathematics Teachers, Teachers College, Columbia University, July 31, 1957.

the mathematical sciences. A wider understanding and appreciation of the role of mathematics in modern society has been set in motion. These, in turn, are cultivating a broader base for the support of mathematical science—financial as well as moral.

These introductory remarks are sufficient to indicate that the new state of mathematical science has many aspects. In the time available it would be impossible to deal with more than one or two of them. So I would like to limit myself to a short sketch of some of the new fields and organizations in the mathematical sciences that have come into existence in the United States during the last quarter of a century. Such a sketch is perhaps as good a method as any for conveying some idea of the character and magnitude of the mathematical expansion now under way.

First, however, I would like to explain the use of a term I have been using quite freely—*mathematical science*. This term is becoming widely used to include all fields of mathematics, both pure and applied. One might question whether this relatively new term in the context we are using it has any advantage over the term *mathematics* or the term *pure and applied mathematics*. I think it does. The term "mathematics" has long carried a connotation toward the fields comprising what we normally think of as pure mathematics. On the other hand, use of the phrase "pure and applied mathematics" tends to conjure up the age-old and fruitless debate as to what constitutes applied mathematics. The newer term "mathematical science" may be relatively safe from such connotations.

What we need is a label that carries with it the concept of a mathematical type of approach to problems in any field of study on which progress can be made through such an approach. Such problems exist not only with respect to positive integers, rational numbers, real numbers, complex numbers, and other collections of abstract elements upon which pure mathematics is based but also in many substantive fields such as quantum mechanics, fluid mechanics, games, logistics, information transmission, high-speed computing, sampling, design of experiments, genetics, epidemics, economic processes, learning processes, and decision making, to mention only a few. In making an effective mathematical attack on any problem or process in any of these substantive fields, the investigator must have enough of a grasp of the substantive aspects of the problem to set up a reasonably faithful mathematical model of the process under study and enough mathematical facility and ingenuity to get practically useful results from the model. The outcome should be new insight into the process that could be obtained either not at all or with great difficulty without using mathematical methods. It is true that the investigator makes liberal use of results that have been obtained from studies in pure mathematics. On the other hand, the mathematical models thus constructed generate fundamental new ideas for studies in pure mathematics. The basic ideas underlying many of the highly developed fields in pure mathematics such as calculus, differential and integral equations, group theory, measure theory, stochastic processes, Fourier transforms—to mention only a few—were generated in this manner.

The term *mathematical science* comes as near to covering the range of concepts I have been discussing as any that could be devised. Similarly, *the mathematical sciences* is an effective label to apply collectively to the different areas, or fields, of mathematical science. These terms have been adopted by several federal government agencies as descriptive of activities falling within their mathematical programs. Thus, there is a Program of Mathematical Sciences in the National Science Foundation; a Section (or Division) of Mathematical Sciences in the Office of Naval Research, in the Army Office of Ord-

nance Research, and in the Air Force Office of Scientific Research. The mathematical organization headed by Professor Richard Courant at New York University is called the Institute of Mathematical Sciences. I believe the term will become generally adopted and used to embrace the whole range of scientific activity of a mathematical nature, whether in the basic fields (pure mathematics) or the substantive fields (applied mathematics).

Now let us turn to the new fields and organizations in the mathematical sciences which have emerged within the last quarter of a century. Twenty-five years ago there were three national mathematical societies in the United States: the American Mathematical Society, which was founded in 1894, the Mathematical Association of America founded in 1915, and the National Council of Mathematics Teachers founded in 1920. These three societies, as we all know, are devoted primarily to research in pure mathematics, college mathematics teaching, and the teaching of mathematics in secondary schools, respectively. In 1932, their memberships were approximately 1800, 2000, and 5000, respectively, while their present memberships are about 5400, 7000, and 11,700. Under the vigorous stimulation of the Society there has been a tremendous amount of research progress in algebra, topology, functional analysis, geometry, and other fields of pure mathematics during the last quarter of a century. the growth rate of mathematical journals and, in particular, of *Mathematical Reviews*, is clear evidence of this. Even the briefest mention of the high lights of the advances in these fields by someone thoroughly familiar with them—if, indeed, such a person exists—would more than fill the time for any talk he could give of reasonable length. As for the activities of the Association and Council, we are all familiar with the work they have done and are continuing to do in the improvement of mathematical education at the college and secondary school levels.

The development of mathematical science up to 1932 had taken place largely in pure mathematics and in the physical sciences and engineering. Since the turn of the century, there had been a particularly rich growth of mathematical analysis of problems in physics in such areas as statistical mechanics, relativity, and quantum mechanics.

By the late twenties and early thirties there were a few persons scattered about who were becoming interested in mathematical analysis of problems in such areas as economics, statistics, psychology, and biology. By 1931, enough interest had developed on the part of mathematicians interested in economics and economists interested in mathematics to establish the Econometric Society to foster the development of mathematical and statistical methods in economics. The Econometric Society is international in organization, although its membership is predominantly American. It has grown steadily and now has about 1600 members. In recent years some of its members have been very active in research on problems in such areas as game theory, linear programming, logistics, and decision making.

During the same period there emerged a small group of mathematicians interested in statistics and statisticians interested in mathematics who decided to organize a society of their own. The result of their efforts was the founding, in 1935, of the Institute of Mathematical Statistics. This group has now grown to about 1800 members. Its members do research on the mathematical aspects of statistical problems on a very wide front. In the pages of its journal we find contributions to the theory of sampling, tests of statistical hypotheses, sequential analysis, decision theory, order statistics, non-parametric statistical inference, analysis of variance models for the design of experiments, and many other areas. Much of the current research in probability theory and stochastic processes is published in the same journal. There is a widespread demand for

mathematical statisticians in universities, government, business, and industry. Sampling experts are in demand in business and government in connection with polls, market research, and other survey operations based on sampling methods. Specialists in the design of experiments are in strong demand in research, development, and testing laboratories in industry and government.

During the same year, 1935, in which those interested in probability and statistics were organizing themselves into the Institute of Mathematical Statistics, a small group of mathematicians interested in the foundations of mathematics and logic and philosophers interested in symbolic logic founded the Association of Symbolic Logic. The Association now has about 500 members. Its members have been very active in research on semantics, axiomatics, quantification theory, and other problems in logic and the foundations of mathematics. The logical theory of computing machines has also occupied the attention of some members of this group in the last few years.

The next organization in the mathematical sciences I would like to mention is the Psychometric Society. It was also organized in 1935 by a small group of mathematically-oriented psychologists and mathematicians who were interested in psychology. It now has a membership of nearly 600. The research activities of its members cover such areas as the mathematical theory of test construction, psychological factor theory, learning theory, psychophysics, and scaling of psychological phenomena. Its members developed the theory and principles which provide the foundation for the vast system of achievement and aptitude examinations now used in the United States for purposes ranging all the way from measuring IQ's of preschool children to testing entrance qualifications of applicants for graduate schools, not to mention personnel selection purposes for jobs in business, industry, and government.

Thus far, only two societies in the mathematical sciences have been established which are concerned primarily with the social sciences. They are the Econometric Society and the Psychometric Society. On the other hand, there are definite trends toward wider use of mathematical methods in other social science fields such as sociology, political science, and anthropology. The Social Science Research Council has sponsored a Committee on Mathematical Training for Social Scientists for the past five years. This Committee has organized institutes for the mathematical training of social scientists during the summers of 1953, 1955, and 1957. The effects of these summer institutes are already becoming evident in various universities of the country in two ways. First, in the form of encouragement for students in the social sciences to take more mathematical training. Second, in an increase, on the part of some of the younger social science researchers, in the use of mathematical approaches to various categories of social science problems that are now showing up in research contributions to social science journals.

Parallel to the movement toward greater use of mathematical methods in the social sciences, there has been a similar development in certain categories of biological problems, particularly in genetics, biological-growth problems including epidemiology, and the design of controlled experiments and analysis of the results. For more than a quarter of a century, papers involving mathematical and statistical attacks on biological problems have been appearing in widely scattered biological journals. However, it was not until 1947 that an independent society was organized for the specific purpose of promoting the development and application of mathematical and statistical methods to biological problems, although a Section of the American Statistical Association had been in existence for this purpose for several years. The group thus organized is the Biometric Society. Like the

Econometric Society it is actually an international society although nearly half of its members are American. It now has about 1400 members.

The period of preparation for World War II, the military research and development demands of the War itself, together with the postwar growth in industrial research, development, and production have all provided a strong stimulus for the introduction and diffusion of mathematical methods into industry and business. The mathematical science organizations which have sprung into existence since the War are dedicated, in the main, to the development and application of mathematical methods for dealing with various categories of problems in industry and business.

The first of these organizations to appear was the American Society for Quality Control which was established in 1946. This is an engineering type of society whose basic purpose, however, is to promote the application of mathematical and statistical methods to problems of establishing and maintaining specified levels of quality of mass-produced articles. Its founders were industrial engineers and statisticians who had been developing probability and sampling methods of controlling quality since the mid-twenties. Some of these methods were widely introduced into the war-production industries during the War under the sponsorship of the War Production Board. They had been found so effective in establishing and maintaining levels of quality in mass-produced military matériel designated by military specifications that the methods were taken over by manufacturing firms and applied to quality-control problems of their peacetime manufacturing. The American Society for Quality Control has an extremely vigorous growth record. At the present time, only eleven years after its founding, it has more than 10,000 members. It has eighty-five local chapters as well as four chapters in Canada, one in Mexico, and one in Japan. In addition to its continuing program of extending standard probability and statistical methods of quality control to all mass-production industries, it is very active at present in stimulating the automation of these methods and spreading modern statistical principles and procedures of the design of experiments in industrial research and development.

Another area of mathematical science to receive the stimulation of the intensified research and development effort of World War II was high speed digital computing. The first relay type high-speed digital computer was developed by the Bell Telephone Laboratories just before the War. The first high-speed electronic digital computer was developed during the War at the Moore School of Electrical Engineering at the University of Pennsylvania for the Ballistic Research Laboratories at the Aberdeen Proving Ground. Since that time there has been a phenomenal development of electronic digital computing machines. This field is now big business. Two areas of mathematical science have played key roles in this development. The first is the logic of automata which is fundamental in the design of electronic computing machines. The second is numerical analysis involved in setting up and programming problems for computation with these high-speed machines. The existence of these machines and the associated science of numerical analysis and programming have made it feasible to deal with problems of computing and data-processing in science, business, industry, and government whose attack could only be dreamed about a few years ago. These will be the brains of the new age of automation we are now entering; mathematicians and electronic engineers will be the brains of the machines.

This new field of mathematical science has been accompanied by the founding of the Association for Computing Machinery in 1947. The members of this group are, of course, mathematicians interested in the mathematical problems of high-speed digital computing and data-processing

and electronic engineers interested in the problems of design and construction of the machines. The Association has a strong growth record and now has over 3000 members.

The next area of mathematical science I wish to mention was literally born during the War. It is called *operations research.* Early in the War a small group of scientists attached to the Fighter Command of the Royal Air Force developed, by experiment and analysis, tactics for radar-equipped fighter planes against Nazi bombers which proved decisive in winning the Battle of Britain in the summer of 1940. The idea of having a small group of scientists study military operations looking toward improvement of tactics and better performance of men and equipment in combat soon spread to the Coastal Command and Bomber Command of the Royal Air Force and to the Admiralty. These groups became known in England as operational research groups. In 1942, similar groups were established in our own Air Force and Navy. They made many studies concerning antisubmarine operations, convoy operations, bombing operations, and fighter operations which resulted in improved tactics and operational procedures. As a matter of fact such groups are still in existence for the purpose of improving strategy, tactics, and performance of equipment and operators on the basis of scientific study of experimental operations and maneuvers. The operations research approach to problems amenable to such an attack is largely a mathematical approach heavily loaded with probability and statistics. Since the War the idea of having operations research groups has spread to business and industry. So at the present time we find operations research groups in many of our large business and industrial corporations whose job is to make scientific studies and recommendations concerning production scheduling, inventory control, distribution, warehousing, plant location, and many other operations problems.

The development of operations research has been accompanied by the organization of the Operations Research Society of America in 1952. Its present membership is over 2000 and includes many mathematicians and statisticians among its members.

Another new society in the mathematical sciences is the Society for Industrial and Applied Mathematics which was established in 1952. It is growing rapidly and now has about 1300 members. The mathematical interests of its members cover a wide range including modern mathematical problems that arise in industrial research and development as well as more classical problems such as those that occur in fluid mechanics.

The most recent society to be formed having strong interests in mathematical science is The Institute of Management Sciences which was organized in 1953 and now has about 1400 members. The Institute is concerned with the application of mathematical and other scientific methods to product pricing, inventory control, production scheduling, renewal or replacement of equipment, and other problems of business and industrial management. Mathematical methods of particular relevance to these problems include high-speed digital computing and data-processing, probability and statistical inference, and linear programing.

In connection with mathematical science as applied to problems in the business world, I should also mention the Society of Actuaries. It was formed in 1949 as a merger of two actuarial societies that had been in existence for a considerable number of years. The Society has nearly 1000 Fellows and over 700 Associates, almost all of whom are actuaries in life insurance companies. Membership in these two categories is gained only by passing a series of examinations. The Society and its members are concerned with all mathematical and statistical aspects of problems related to the life insurance business.

With all the interest and growth in the

mathematical sciences during and immediately after the War the question arose as to whether it would not be well to set up a division in the National Research Council to deal with the common problems of the various mathematical sciences at the national level. Up to this time the interests of mathematics had been served in the Division of the Physical Sciences of the National Research Council. A committee was established in the Division of the Physical Sciences to study the problem of the representation of the mathematical sciences in the National Research Council. On the basis of recommendations of this Committee a division now known as the Division of Mathematics was established in 1950 in the National Research Council. The membership of the Division consists of fifteen representatives from most of the societies I have referred to earlier, six members-at-large, and six liaison members from as many government agencies. The work of the Division, which is carried on largely by committees, covers a wide range of activities including recommendations for fellowships and grants financed by the National Science Foundation and other government agencies, organization of conferences in the mathematical sciences, promoting the regional development of mathematical research, maintaining contact with the International Mathematical Union, management of revolving funds for publication of mathematical books and tables, and advisory service on mathematical matters for federal agencies. The Division is still young. It will be playing an increasingly important role in the future in establishing and maintaining the mathematical sciences in the scientific and technological life of the nation.

Now let us take a quick look at the financial support of the mathematical sciences and mathematical education. The widespread interest in and appreciation of the importance of the mathematical sciences which has emerged since World War II has been accompanied by a growing financial support of education and research in the mathematical sciences. During the current fiscal year more than six million dollars has been made available for the support of research in the mathematical sciences by the three research-supporting military agencies, namely, the Office of Naval Research, the Army Office of Ordnance Research, and the Air Force Office of Scientific Research. Over one million dollars has been provided for research and approximately 100 predoctoral and postdoctoral fellowships have been awarded in the mathematical sciences by the National Science Foundation during the same period. Fellowships are being granted in the mathematical sciences every year by the Guggenheim Foundation, the Sloan Foundation, and other private organizations.

On the side of mathematical education, the National Science Foundation, during the summer of 1957, is supporting thirteen summer institutes in mathematical education for secondary school teachers, one for college teachers, and one for both categories of teachers. The Carnegie Corporation has spent more than a half-million dollars in the last two years in the field of mathematical education at the secondary school level, of which $150,000 was granted to the College Entrance Examination Board Commission on Mathematics to use in connection with its work toward the revision and modernization of the secondary school mathematics curriculum. The Ford Foundation has granted to the Social Science Research Council nearly a quarter of a million dollars for the support of its summer institutes for mathematical training of social scientists in 1953, 1955, and 1957.

In conclusion there can be no doubt that mathematical science is entering an exciting new age. Great strides are being made in research in the fields of pure mathematics. But equally significant is the appearance of new fields of mathematical science which, within the last twenty-five years, has culminated in the

organization of nearly a dozen new societies, most of which are still in vigorous growth stages. These societies are dedicated to the development and application of mathematical methods for dealing with problems of many different kinds that arise in all fields of science, industry, business, and government. There are new sources of interest for the social sciences in the possibilities of mathematical methods in dealing with their problems.

This growth has been accompanied by a great broadening of appreciation for the role of mathematical science in modern technological society. It is bringing many new problems in its wake. On the educational side it involves restudy of the mathematics curriculum at the secondary and college levels and improvement of procedures for recruiting and training teachers at these levels. On the research side it involves problems of recruiting an adequate number of people with mathematical talent to meet the mounting demands for mathematical scientists. In the case of those who do not choose to pursue pure mathematics, questions arise as to how best to provide training in mathematics to enable them to make mathematical attacks on problems in their area of interest.

The result of this new movement will be the institution of the mathematical sciences broadly into the scheme of things in much the same way that the biological, chemical, geological, and physical sciences have now become established.

What's new?

BOOKS

Secondary

An Introduction to Euclidean Geometry, J. C. Eaves and A. J. Robinson, Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1957. Paper, xii+327 pp., $4.25.

The New Applied Mathematics (fifth edition), Sidney J. Lasley and Myrtle F. Mudd, Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1958. Cloth, 457 pp., $3.48.

Plane Geometry (with Teacher's Manual), John F. Schacht and Roderick C. McLennan, New York: Henry Holt and Company, 1957. Cloth, xvi+494 pp., $3.84.

College

An Analytical Calculus (Volume IV), E. A. Maxwell, New York: Cambridge University Press, 1957. Cloth, ix+288 pp., $4.00.

Economic Models, E. F. Beach, New York: John Wiley and Sons, Inc., 1957. Cloth, xi+227 pp., $7.50.

Numerical Analysis, Kaiser S. Kunz, New York: McGraw-Hill Book Company, Inc., 1957. Cloth, xv+381 pp., $8.00.

INSTRUCTIONAL MATERIALS

Designing the Mathematics Classroom, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C. 40-page illustrated booklet written by Lawrence P. Bartnick, $1.00 (postpaid if remittance sent with order).

Improvement of the Teaching of Mathematics, F. R. Born, State Department of Education, Oklahoma City 5, Oklahoma. 94-page illustrated booklet prepared by the Mathematics Workshop at Oklahoma State University, held at Stillwater in the summer of 1957, 50 cents.

Mathematics Power Test, E. A. Habel, 741 West Mallory, Pensacola, Florida. Test from *How to Do Mathematics to Eliminate Errors* by E. A. Habel, sample copy free.

Your Opportunities in Science and Engineering, and

Your Opportunities in Industry as a Technician, Education Department, National Association of Manufacturers, 2 East 48th Street, New York 17, New York. Illustrated booklets, each available in reasonable quantities free and postpaid.

The "make 'em run" theory

RAY BENGTSON, *Ramsey Junior High School, Minneapolis, Minnesota.*

A teacher of some experience sets forth his ideas on how to do the most for students with superior ability.

FOR THE PAST few years I have taught elementary algebra to a special class of superior ninth-grade students. One observation intrigues me more and more as I work with these youngsters. In every group I have pupils who suddenly burst into a new level of ability, drive, and stamina. I am trying to find why this happens—and how to generate it. By now I have some theories.

I doubt that added maturity is an important factor because there is a tendency for several students to make the jump at the same time, and it is improbable that they all matured simultaneously. Motivational factors account for changes in some of the students, but others were already willing to work. In fact, they were eager and ready for action when they came in on the first day of school.

Some of my students, it seems to me, have licked an extra handicap–one which grew *because* they were gifted. Results had come to them so easily that they could be leaders with little effort. After years of this they didn't know how to open up and use all their power. Either they had forgotten how to run or they never had learned what running is. Finally they did learn, and the change was obvious.

After several years of wrestling with this problem I have an approach based on the "make 'em run" theory. This says, "If you can just get a kid to run fast enough, he'll know what running is." This sounds reasonable but bypasses the main problem—how do you get them to run? Well, let's move over for a while to the "make 'em walk" corollary. This scientific oddity is a generalization based on one observation.

My young nephew could walk if he kept one hand on the furniture, and stood alone only when reaching from one piece to another. One day he got off course, tottered across the center of the room, and safely reached the far side. A strange expression crossed his face when he looked back and realized what he had done. He was delighted with his discovery and practiced almost continuously.

Let's consider this situation. The child had potential he was not using *although he had plenty of desire.* The situation started normally and then led into one which called for a new ability. He was too busy to be aware of his new ability at the time it came to him. Excitement helped him in calling forth unsuspected reserves. He succeeded by accident—but sometimes parents arrange this accident!

Can this pattern be used to teach pupils to "run"? I think so. And I believe it also works on people who aren't as interested as they should be. Running is pretty fascinating stuff, and it can get to be a habit. Here are some "make 'em run" items which proved particularly effective in my classes.

The first was a lecture by a university mathematics professor. His subject was famous unsolved problems in mathematics—things like the four-color problem, Fermat's Last Theorem, and many others. The presentation used fairly elementary language but the ideas were college level. It was rugged! The pupils were

successively stunned, incredulous, fascinated, and finally engrossed in a flood of ideas. No one understood the entire lecture but most of them got the gist of all of it. (They were too busy to realize that they were running.)

Another item was a two-day unit entitled "Is There a Physical Fourth Dimension, or How to Change Left Shoes into Right Shoes." We developed ideas about the nature of a world with four physical dimensions and whether such a world might exist. We never did manage to visualize that shoe on its trip into the fourth dimension and back out again. All we could do was develop logically what the result of such a trip might be. And no one really understood a four-dimensional supercube. Common sense was sometimes a hindrance; we were busy as men from "Flatland" trying to play leap frog.

"Storming" helps in any of my classes. I use fairly hard problems: the proper level of difficulty is important. The students try to solve these at a rate limited only by the speed with which they can write them down. It is quite a shock when an "A" student learns that his neighbor goes three times as fast and is just as accurate. I may be sticking my chin out, but I believe students can be *too* concerned with neatness and complete accuracy. Sometimes this is a plausible excuse for living at half speed. Emotional problems may keep the student from cutting loose. We try for speed with reasonable accuracy instead of accuracy with reasonable speed. The problems in storming are similar to those in learning to play a very fast piece of music. You can't develop speed in either without making some mistakes in the early stages.

I don't mean to give the impression that all "running" has a lot of surface activity. Sometimes we proceed by giant steps instead of scurrying with many small steps. One variation on storming is to attack very difficult problems. For example, I use some advanced algebra problems at the end of my factoring unit. This accents basic principles of factoring and forces students to THINK.

All of these experiences seem to have certain common elements. Each has a start the pupils can manage. Each gets into problems that can be managed only by stretching—and then by stretching even further. It is essential that the student be excited or at least thoroughly interested—so must the teacher! (I haven't yet found a self-sufficient unit.)

Getting students to use their full potential is a difficult part of teaching. It may be our most important responsibility. The achievement in my classes, as measured by nationally standardized tests, has improved since I started to apply my "make 'em run" theories. This is true despite the fact that many of our activities deal with topics not touched by the conventional tests. The extra time and effort involved seem to me to be worthwhile.

The usual mathematical symbols for "less than" and "more than" are very closely related to the symbols used for indicating "crescendo" and "diminuendo" in writing music. This does not seem to be an accident since until quite recently in the history of both of these arts, music was generally considered to be a branch of mathematics.—*Frank Hawthorne, Supervisor of Mathematics Education, New York State.*

Trisecting an angle

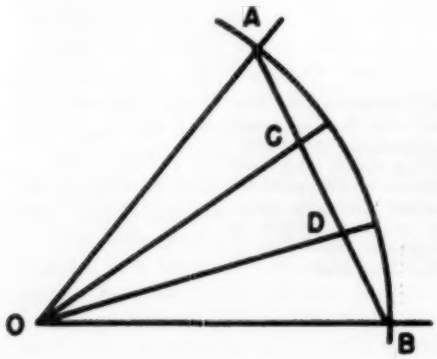
HALE PICKETT, *El Camino Junior College, El Camino College, California.*

—A convincing answer to an often-heard question.

THE PROBLEM OF TRISECTING an angle leads to the following difficulties. When a class is told that trisecting an angle is impossible by means of compasses and an unmarked straight edge they accept the fact for the present, but just as soon as a teacher demonstrates how to trisect a straight line, some would-be mathematician gives the following solution for the problem.

Draw arc AB, then draw chord AB. Trisect chord AB at points C and D, then draw OC and OD, which seems to the student to be a solution for trisecting an angle. It appears without proof to the student that the lines which trisect the chord also trisect the arc. This is not true and it is difficult to disprove, if one is not prepared.

Figure 1



I suggest that at first an intuitive proof be given. Take an obtuse angle since it works better.

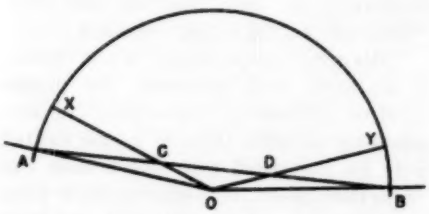


Figure 2

Trisect the chord AB at points C and D, then draw XCO and YDO. By observation it appears that $\angle AOX = \angle BOY$; however, it seems equally obvious that both these angles are less than $\angle XOY$, especially if we compare arc AX and arc BY with arc XY.

The geometric proof that

$$\angle XOY \neq \angle YOB$$

depends upon an indirect proof, which again gives rise to a new difficulty, unless the steps of an indirect proof have been developed. And in nearly all cases this proof is deterred by the fact that a theorem needed in the proof has not been developed. (The bisector of an interior angle of a triangle divides the opposite side into segments that are proportional to the adjacent sides.)

Referring to Figure 2:

Given: $AC = CD = DB$

To prove: $< YOX \neq < BOY$

Proof

Either $\angle BOY = \angle YOX$ or $\angle BOY \neq \angle YOX$

Assume $\angle BOY = \angle YOX$

In triangle BOD and triangle DOC

$$\frac{OB}{BD} = \frac{OC}{CD}$$ The bisector of an interior angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides

Substituting: $CD = BD$ Hypothesis

$$\frac{OB}{CD} = \frac{OC}{CD}$$

Then $OB = OC$ (Multiplying by CD)

This is absurd since OB is a radius and cannot equal OC which is only a part of a radius.

Therefore $< BOY \neq < YOX$ Having proved one of two opposite statements false, then we accept by indirect proof the other as true.

Purdue-General Electric Fellowship Program

Fifty all-expense summer fellowships for junior and senior high school teachers of mathematics will be awarded by the Department of Mathematics and Statistics of Purdue University. These fellowships were made available through a grant by the General Electric Educational and Charitable Fund. The courses in the six-week program—June 23 to August 2, 1958—carry eight semester hours of graduate credit and have been specifically designed to increase the teachers' breadth and depth of knowledge of mathematics in areas relevant to the courses they teach. This program is especially for the classroom teacher, so that in making the awards consideration will be given to potential leadership as a teacher and counsellor, and interest in professional improvement. To qualify, teachers must have a bachelor's degree, have completed a course in calculus, and teach in Illinois, Indiana, Iowa, Kentucky, Michigan, Minnesota, Missouri, Ohio, Tennessee, West Virginia, or Wisconsin. Further information may be obtained from the Purdue-General Electric Fellowship Program, Department of Mathematics and Statistics, Purdue University, Lafayette, Indiana.

Things and un-things

W. W. SAWYER, *University of Illinois, Urbana, Illinois.*

Cats and un-cats; dogs and un-dogs. Can you use this to teach $(-3)\times(-2)=+6$*?*

MOST WAYS OF EXPLAINING negative numbers are a bit forced at some point. A scientifically sound explanation can be based on directed numbers, with $+3$ related to "3 units eastward" and -3 to "3 units westward." This works very well for addition and subtraction, but not at all naturally for multiplication and division.

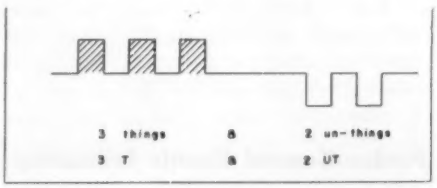
In other explanations, one is forced to interpret a minus sign in two different ways. If -3 is related to "a bill for 3 dollars," one has somehow to make $(-5)\times(-3)$ into "tear up 5 bills for 3 dollars." Pupils may well feel there is some trickery going on, when we choose the meaning to suit ourselves.

Now, of course, since we cannot in this world actually have -3 apples, there must be some sense of unreality connected with negative numbers. I am about to describe a universe in which you could have -3 apples. This universe is a little fantastic, but, if anything, that should commend it to children. Once we have this universe imagined, there is no cheating. In it, children could learn to add and multiply negative numbers with the same apparatus and procedures that actual children use in Grade 1 for actual numbers.

Grade 1 arithmetic is really a series of experiments in physics. You put two apples and two apples together and you find you have four apples. Or, starting with four apples, you can break them up into two apples and two apples.

I will write in the form $2T\ \&\ 2T\leftrightarrow 4T$ the fact of physics that two things and two things can be combined into four things, and vice versa. Children are thus led to write the arithmetical statement $2+2=4$.

We now make our break with the accepted physics, and suppose the world contains not only things but also un-things. If an un-thing and a thing meet, they wipe each other out. An un-apple wipes out an apple, an un-dog wipes out a dog, and so on. It can work the other way round, too. If you have a box with nothing in it, a little later you may find it contains a tiger and an un-tiger, or a dollar and an un-dollar. (This latter fact raises special problems in connection with forgery.)



The illustration above shows 3 things and 2 un-things. An un-thing is written UT for short. 3 things and 2 un-things can change, reversibly, into 2 things and 1 un-thing, or simply into 1 thing. In symbols

$$3T\ \&\ 2UT\leftrightarrow 2T\ \&\ 1UT\leftrightarrow 1T$$

USE OF THE SYMBOL *U*

If X stands for some collection of objects, UX stands for what would wipe

that collection out. Thus, if D stands for dog, and C for cat, $U(C \,\&\, D)$ stands for what would wipe out a cat and a dog. Clearly, that is an un-cat and an un-dog, i.e., $UC \,\&\, UD$. So $U(C \,\&\, D) = UC \,\&\, UD$. U obeys the distributive law.

Now, what about $U(UC)$? This means something that wipes out an un-cat. But that is a cat; for when a cat and an un-cat meet, they combine to form emptiness. So $U(UC) = C$.

Un-seven

The answer to the question, "How many?" is a number. Suppose we are asked, "How many cats in this room?" and there are in fact seven un-cats in the room. I suggest that the answer should be "Un-seven." We may write this number $u7$. So "seven un-cats" and "un-seven cats" are simply two ways of saying the same thing.

$$7UC = u7C$$

Adding and subtracting

Adding and subtracting can now be done in the usual way. For example, what is $3+u5$? Get 3 cats and 5 un-cats. Three of the un-cats will wipe out the three cats, and in no time you will be left with two un-cats.

$$3C \,\&\, 5UC \rightarrow 2UC$$

So

$$3+u5 = u2. \quad \text{Answer, un-two.}$$

For subtraction; what is $3-u5$? You have 3 cats and you want to give someone 5 un-cats; what will you have left? Well, take your 3 cats and stir them up a bit; with luck they will change into 8 cats and 5 un-cats. Give away the 5 un-cats. You are left with 8 cats. Answer, eight. Alternative method: ask, "Un-five cats and what makes three cats?"

Multiplication

This, too, is done in the usual way. If we want to find what 3×2 is, we first take two things, TT. Then we take three sets of two things, $TT\ TT\ TT$. This gives us six things.

As

$$3 \cdot (2T) \leftrightarrow 6T,$$

we say

$$3\times2 = 6.$$

Exactly the same method will work if we want $(u3)\times(u2)$. We must form un-three sets of un-two things each. Un-two things is the same as two un-things (see paragraph headed "Un-seven"). So un-two things is the set $UT \,\&\, UT$. We want un-three such sets. This is the same as three un-sets. The un-set is $T \,\&\, T$, or $2T$. Three of these give $6T$. Thus, $(u3)\cdot(u2T) = 6T$, and we say $(u3)\times(u2) = 6$. One could also carry through this argument by using the kind of diagram shown earlier, with things as mountains and un-things as valleys into which the mountains may fall. This would probably be much easier to follow.

$3\times(u2)$ and $(u3)\times2$ can be done in the same way.

It only remains to identify $u3$ with the usual sign -3. Since 3 things and un-three things combine to give emptiness, it is clear that $3+(u3) = 0$. Thus, $u3 = 0-3$; $u3$ is the result of subtracting 3 from zero. This is ordinarily called -3.

The use of u allowed us to distinguish between un-three (negative 3; -3) and "subtract 3." Un-three cats is an object. "Take away three cats" is an order.

Un-things and electrons

As a concluding note it may be pointed out that un-things are perhaps not so fantastic after all. In Dirac's theory of electrons, there were things called "electron-holes" which, when they met electrons, destroyed the electrons and themselves ceased to be. Of course, this annihilation of matter produced considerable radiation. If one ignored the radiation effects, one would have an actual physical model for things and un-things.

Elementary school and high school mathematics for the colleges?

M. F. SMILEY, *State University of Iowa, Iowa City, Iowa.*

Today's pressure on college enrollments has forced many colleges to adjust courses much as high schools have done in the past. Can this be justified?

THE PURPOSE OF THIS NOTE is to present the author's feelings about the sort of *college* general education course in mathematics described in a recent paper[1] in this journal. In (S) one finds the following outline for such a course:

1. The concept of number, its growth and development
2. Our system of notation
3. The symbols of mathematics and their meanings
4. Inexact mathematics (*sic*)
5. Percentage and its uses
6. Installment buying and planned savings
7. Making and using graphs
8. The application of geometry to measurement.[2]

We object to none of these topics as part of a general education, although the term "inexact mathematics" is not a happy choice. However, we feel that every one of these topics should be required of *all* students somewhere between the fourth and tenth grades. It is our thesis that college courses of this nature are doing very serious harm to the true goals of general education in America.

For the last ten years, a course very like that described in (S) has been required for the liberal arts degree in at least one large state-supported university. About one-half of the freshmen are excused from this requirement on the basis of an examination, and thus are automatically freed of any requirement in mathematics for the liberal arts degree. Nonetheless, in the fall of 1956, about 900 freshmen were enrolled in the course. There seems to be no relation between the number of semesters of high school mathematics taken by the student and his success on the excusing examination.

This course was designed in just the way prescribed[3] in (S).

Since one can so easily become entangled in the snares of equivocation[4] in the abstract discussion of education, let us bypass such discussion and turn to a consideration of the effects that the *college* course described in (S) has had on the efforts of the students and teachers in the elementary and the high schools.

Why should elementary and high school teachers, overburdened with the ever-increasing demands placed on them by

[1] T. M. Simpson, "Mathematics in the college general education program," THE MATHEMATICS TEACHER, L (1957), 155–159. We shall refer to this paper as (S).

[2] *Op. cit.*, p. 158.

[3] A colleague succinctly describes this method as follows: "You go around and ask people who don't use mathematics what mathematics they use."

[4] For example, contrast the statement of Hutchins, (S), pp. 158–159, with the definition of general education given in (S), p. 155.

modern society, continue to exert their efforts to perform a job which will be taken care of later by another state-supported institution? That so many elementary and high school teachers have carried on their duties to teach the material listed in (S) (and have, of course, done *much* more) seems to me to indicate a dedication to their profession which is little short of heroic. That some students and teachers have adopted the motto, "Let the colleges do it," is surely natural enough, *so long as the colleges will.*

Everyone nowadays is aware of the vital need to increase the *learning* of mathematics, both in quantity and in quality. The National Science Foundation is at present engaged in something very near to a "crash program" of summer (and winter) institutes designed to foster increased study of science, and particularly of mathematics, *in the high school.* I submit that it is the duty of the colleges to support this effort by ceasing to teach elementary and high school general mathematics.

Have you read?

HACH, ALICE M. "Add Interest to Junior High Mathematics," *NEA Journal*, October, 1957. pp. 433–434.

Miss Hach has pointed her finger at an area of mathematics teaching that has had too little attention, namely the junior high school. She does not feel that every student here is a potential scientist, but that those who are should not be turned away by boredom. Among the suggestions she gives to whet their appetites are: use advanced topics to give drill in previous areas, encourage neatness which in turn develops pride, help the students in self-evaluation, provide an interesting classroom, strengthen mathematics of the elementary school by helping students recall what they learned there, use field work and the laboratory. This is an article you will want to read; it is packed with ideas.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

PARSONS, G. L. "Teaching the Teacher," *The Mathematical Gazette*, February 1947, pp. 1–8.

This is a presidential address to the Mathematical Association of Great Britain in 1956. It is a summary of some of the issues appearing before the association through the years of its existence.

Among these is the problem of geometry. This problem has many facets, among them, that Euclid as such tends to become ossified, that construction and theorems are separate problems, that applications and abstract geometry should be separated, that sequence is unimportant, that freedom to experiment is necessary, and that geometry should be more three dimensional.

A second idea Mr. Parsons offers is "Unity-in-Variety" of mathematics. This means non-compartmentalized mathematics. This has led to the introduction of mathematics to more people, but has consequent loss of intensity overshadowed the gains? Has the presentation been reduced to the historical? Textbooks and teachers have fallen short in this area.

A third problem is that of the content which has been dropped from the curriculum, but that the student is somehow expected to know. Mr. Parsons presents many other ideas and, although his comments are directed to mathematics in Great Britain, they offer much for us to think about.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

VAN DOREN, CHARLES. "Junk Wins T.V. Quiz Shows," *Life*, September 23, 1957, pp. 137–148.

This is not an article about mathematics but it does present a good philosophy of education and, since your students probably remember the author, they may enjoy reading the article. They will be interested in how he happened to enter the quiz show but they will be more surprised at his reactions.

For example, that the knowledge quiz show contestants exhibit is only junk, that it is advantageous to forget things, that knowledge must bring joy and continuing learning. Knowledge implies civilization. Wise men know the connections between the things they know. Knowledge is more concerned with the unknown than the known. Quiz shows have the answers, while in education there are no answers. An educated man and a quiz show contestant are moving rapidly in opposite directions. The world of the educated man is foggy and dark with unlighted passages, but that of the contestant is a bright circle of light where all is certainty. You will be interested in this article.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Mathematics in South Australia

ALLEN WOOD, *Willunga Higher Primary School, South Australia.*

A visiting teacher in the Wauwatosa, Wisconsin schools tells us about Australian mathematics and Australian educational problems

THE SOUTH AUSTRALIAN education system, in common with those of all Australian states, differs in many ways from those generally common to the United States, so that any discussion of mathematics teaching and curricula must be preceded by a short description of that system. South Australia has an area approximating one seventh of that of the United States, but has a population of only 850,000, of whom 62% live in the capital city, Adelaide. In its sparsely-peopled areas, small communities have always looked to the state government to develop their railways, roads, water supplies, and power, so that it was natural for the state to take over the responsibility for education, rather than leave it in the care of local boards, as is the case in the United States. A centralized authority located in Adelaide has as its head a Director of Education who is responsible to the Minister of Education, a member of the State Cabinet. The state government provides all moneys so that no local taxes are levied for education. Two teachers' colleges, which pay and train teachers for periods of 1 to 4 years after leaving high school, are under the control of the Education Department, but the academic training program is carried out by the University of Adelaide, where students attend as undergraduates in the different faculties. Teachers may be appointed to any school in the state and are moved wherever the Education Department needs them.

Legislation on the statute books states that the school-leaving age is 15 but it has never been proclaimed. Therefore, children are free to leave school at 14, and many of them do, taking jobs in industry and offices. Under the improved economic conditions prevailing today there is a definite tendency for parents to keep children at school to a later age. Our educational system, traditionally based on the English and Scottish systems but greatly modified to suit our peculiar conditions, is free through 7 years of elementary and 5 years of high school, but nonresidential universities situated in each capital city charge fees. However, there are many scholarships based upon ability and a means test which allow our more able students to take advantage of a tertiary education.

During the last 30 or 40 years, South Australian education has passed through a revolution. At the end of World War I, elementary education was, by law, compulsory and universal; but secondary education was to a large extent confined to a small portion of the population between the ages of 14 and 17. Since 1920 there has been a vast expansion in secondary education, with a conscientious attempt to bring the opportunity for a secondary education to every boy and girl in our community and to encourage them to continue at school as long as possible. Formerly, we were faced with the problem of providing an education at the secondary level for the few who were to proceed to the university or to some specialized vocational position, and the influence of the classical concept of the English grammar

schools was very apparent in the curriculum and, in particular, the mathematics of our high schools.

With these new concepts of secondary education, diverse types of secondary schools have grown up to meet special needs or to conform to special ideas which have been brought forward. Mathematics early became a very important part of education in these schools. But now the problem arises of providing suitable mathematics courses for all pupils, able and less able, who attend at least until they are 15. The existing syllabus appears adequate and satisfactory for the able, but above the reach of others; and the attempts at modifying the earlier syllabus have not been entirely satisfactory. Perhaps the general mathematics courses offered in the United States can be of assistance to us.

Children commence school at the age of 5+ and remain in what is termed the infant school for two to three years before transferring to primary schools at the grade III level. At the end of the seventh grade or about the age of 12+ they are given a Progress Certificate and then proceed to a secondary school, which would be either a high school, junior technical school, higher primary school, area school, or private college. These private colleges, which correspond to the public schools of England and not to the American liberal colleges, charge fees and are not under the control of the South Australian Education Department. Area and higher primary schools are consolidated country schools and take care of the education of the children through elementary and secondary grades, adapting their individual programs to suit the needs of their districts. Tertiary education is gained at the University of Adelaide, which like all Australian universities, is nonresident.

Arithmetic is taught in the primary grades and, by and large, is anything but a popular subject. There are teachers who like teaching it, and pupils who enjoy learning it, but they are in the minority. We are constantly making concessions to the difficulty of the subject, so that, for example, operations which were taught some years ago at the Grade IV level are now taught at the Grade V or even the Grade VI level. This subtraction from the curriculum of the subject matter taught was accompanied by a diminution in the number of lessons given to the subject, but it would be true to state that the amount of time spent on arithmetic in South Australia is still greater than that in American schools. This postponement of many processes and the deletion of others can, in part, be attributed to American influence, particularly the Winnetka experiments in optimum ages. The net result is that there is in the present course a body of subject matter somewhat similar to that included in most curricula in the United States, but with an earlier beginning in abstract conceptions. In addition to American arithmetic is the considerable amount of work necessary in mastering the four processes with our awkward money system containing two halfpence in one penny, twelve pence in a shilling, and 20 shillings in a pound, or with our equally ridiculous weights and measures. Such work is a considerable part of Grade IV and V arithmetic and makes percentage, interest, etc., more difficult in computation in Grades VI and VII than the easier American system.

The minimum course to be followed in all schools is set out in a publication, *The South Australian Course of Instruction.* Books written to cover this course are very often followed rather slavishly by teachers. Unfortunately, there are no complementary workbooks or teachers' handbooks. Arithmetic books are kept as low in cost as possible, so that colour and illustration are not good, and paper, covers, setting, and type faces cannot compare with the American texts. Books are built on the unit plan, are graded in difficulty, and have more difficult problems for the abler student. In most schools regular arithmetic tests are given at the end of

each term by inspectors on their annual inspection of the school. Results, of course, vary, but a 75% class average is considered satisfactory. Many teachers consider that such a result is far too low, and that we should revise our concept of reasonable mastery.

A committee of teachers and administrators, which has currently completed a survey of teaching methods, is issuing a handbook of arithmetic teaching to all elementary schoolteachers, suggesting what it considers to be the most desirable methods. In the past we have patched and eked out the material that has for generations been part of established practice; we have trusted chiefly to tradition, but this and other communities are endeavoring to make certain that we no longer talk about general guiding principles, but are guided by them.

There are different types of secondary schools in South Australia but a majority of children attend high schools or technical schools. In the high schools the education is more academic than that given in technical schools, which cater more to students who do not intend to seek a university education and who are often less able.

Mathematics is such an important section of the high school curriculum that approximately one fourth of the school time is devoted to its study. Unlike the approach in America of taking arithmetic in Grade VIII, algebra in Grade IX, geometry in Grade X, algebra in Grade XI, and trigonometry and solid geometry in Grade XII, in South Australia full mathematics includes arithmetic, algebra, and geometry each year until Grade X, when trigonometry is introduced. By the end of Grade X, the children will have covered, among other things, the mensuration of prisms, cylinders, spheres, pyramids, and cones; compound proportion; compound interest; profit and loss; percentage; rate and time problems; and the use of logarithms in arithmetical computation as part of the arithmetic section. In algebra they will have learned to factor; use directed numbers; draw graphs of linear and quadratic functions; solve linear, simultaneous, and quadratic equations; and gained an idea of maxima and minima values. In geometry, which is approached through practical plane geometry instead of through Euclidean axioms, postulates, theorems, etc., the children will have worked with congruency of triangles, parallelograms, circles, tangents, polygons, and the idea of a locus.

This curriculum is set out by the Public Examinations Board of the University of Adelaide, a board consisting of representatives of the secondary teachers as well as the mathematics faculty of the University, which sets and conducts an external examination for students at the tenth-grade level. To gain a certificate in this Intermediate Examination, as it is called, a student must pass in at least five subjects, of which English must be one. Mathematics, which consists of two examinations of two hours each, counts as two subjects.

All boys and approximately 60% of the girls attending high school take mathematics for three years. Of those taking the Intermediate Examination, usually about 66% pass mathematics. It is at this stage that a large proportion of those attending secondary schools drop out of school to begin earning their own living. Those left are generally aiming at a university education; and most boys, and approximately 60% of the girls, will continue with mathematics, which will occupy almost one third of the school day. Most of the children with low IQs will have left school, so the median IQ of the children of this Leaving Year, as it is called, is about 118 compared with that of 110 in the Intermediate group. This Leaving Mathematics course includes indices, logarithms, projections, maxima and minima, more difficult mensuration, use of logarithms in compound interest and annuities, progressions, inequalities, variation, centroid, orthocenter, similar figures, co-ordinate geometry, and trigonometry. Students can matricu-

late to the University at this stage, but if a student wishes to do university mathematics he must take an extra examination known as special mathematics, requiring additional material.

The final year at high school is known as the Leaving Honours, a pass in mathematics at this level being sufficient for a student to be eligible to commence one of the several first-year university mathematics courses. These are not courses for students with varying levels of attainment, but follow varying branches of mathematics. In this Leaving Honours year calculus, integration, probabilities, and discreteness are among the subjects covered. The study of solid geometry has been practically abandoned in South Australian high schools and there is very little, if any, applied mathematics.

For those in high schools, particularly girls taking commercial courses, who do not wish to take the full mathematics course, a one-subject mathematics course known as arithmetic is available. This includes similar arithmetic to that in the full course, but with the addition of stocks and shares, bills of exchange, taxes, and foreign exchange. There is far less algebra and a very small amount of geometry, almost all constructional work. Stress is placed on short questions, speed and accuracy practice, and quick computations.

Children who attend the technical schools, which perhaps could be compared with English secondary modern schools, generally do so because of the curriculum adopted. Subjects include English, mathematics, social studies, general science, home economics, industrial arts, and art. It is generally true to say that the median IQ in such schools is lower than that in high schools, but many bright children do attend them and less able children attend high schools, for parents have freedom of choice as to which school they will send their children. Such schools have only four years of secondary work, generally do not prepare students for the public examinations, give their own internal certificates, and suit their curricula to the needs and abilities of their students, so that the mathematics course is not as wide in scope nor does it occupy so great a part of the available time as the high school full mathematics course, but all students take mathematics—it is a basic study.

A new mathematics course designed especially for these children by a committee of mathematics teachers will consist of general mathematics involving mechanical operations; industrial and social everyday arithmetic; algebra, involving notation, tables, graphs, formulae, simple equations and problems; geometry, using measurement, scale drawing, interpretation of plans, and notion of similarity; and some numerical trigonometry. The treatment of the subject will (1) be mental, when there will be no recourse to pencil and paper, (2) require speed and accuracy and little written work, (3) use field work when the class will get its data from actual experiment and not from a textbook, thus adding purpose to the calculation, (4) be practical in application, (5) use projects, (6) wherever possible refer to the development of the topic through history, (7) use drill in manipulative work necessary to use the principles taught, (8) revise work, (9) use models, lecturettes, puzzles, etc., so that students will learn through the concrete rather than the abstract approach. Although appreciation and practical uses of mathematics will be the chief aims of the course, its value as a mental discipline is not to be neglected. What manipulation is done will be rigorously taught, but the teacher should be prepared to motivate by showing the purpose of what is taught. There will be no textbook, for it is thought that the spirit of the course will be missed if too much reliance is placed on texts.

South Australia is in a position similar to that of the United States in regard to teacher supply and buildings. The school population is increasing at a rapid rate, and the increase is now being felt in sec-

ondary schools. By using prefabricated units suitable for the climate and placing them in existing schools as well as building new schools, not as elaborate as some in America, but modern and satisfactory, the physical needs are being met. More teachers need to be trained, and too few have been entering the profession, so that the experienced mathematics teachers are being used for the more advanced work, and teachers with no further experience in mathematics than the fourth year of high school are being used to teach the students in the lower secondary grades. While many of these are fine teachers, their lack of a mathematics background must result in less effective work. Unfortunately, the position seems likely to deteriorate further, for, by 1960, enrollments will be almost double those of today. The experiment to be introduced next year in the United States—that of giving scholarships to mathematics teachers to attend colleges and universities for a year—might well be emulated in South Australia if we could find teachers to replace those taken from the classroom during that time.

South Australian problems in mathematics teaching, particularly those of teacher supply and the necessity of adjusting curricula to suit the able as well as the average, have much in common with those of the United States. The efforts made in America to meet them will be of very great interest to South Australian teachers.

Are there several kinds of variables?

The student in this country can play a decisive role in shaping the future frame of mathematics by insisting on a perfect clarification of mathematical concepts and procedures. He must not rest till he receives answers to all questions that bother him.

Have I not heard, he wonders, that $x=2$ is the solution of the equation $x+1=3$, whereas now I am told that $x=2$ is an equation—the equation of a straight line?

How can it be that $x=2$ and $y=2$ are totally different straight lines, whereas $x+1=3$ and $y+1=3$ are essentially the same equation?

Certainly these examples would not involve—no one ever spoke to me about that possibility—two kinds of x and y?

Is there a difference between $x+1=1+x$ and $y+1=1+y$? Are the functions $x+1$ and $y+1$ equal?

Why is $x+1$ a non-constant function and $5+1$ is not, whereas from $x+1=1+x$ it follows that $5+1=1+5$? I could not possibly be up against still another x?

No. In all these cases, my text calls x a variable, just as c is always a constant, a definite number. Or is it?

Which definite number is the constant c in the saying that $x+c$ is a linear polynomial? True, in this case the text says that c is an arbitrary constant. But why does it not say the same about x in $x+1=1+x$? And haven't I read $c+1=1+c$?

Is not $x+2y=2y+x$ the same as $y+2x=2x+y$, which shows that variables may be shuffled? But for variables connected with circles is not $a=\pi r^2$ true and yet $r=\pi a^2$ false? And are not $y=x^2$ and $x=y^2$ different parabolas?

Again, what is the difference between the class of the pairs of numbers (x,y) such that $y=x^2$, and the class of the pairs (y,x) such that $x=y^2$?

There could not possibly exist several kinds of variables, too, without anyone telling me?—*Karl Menger:* Basic Concepts of Mathematics.

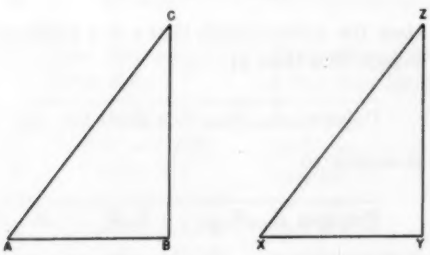
The use of congruence in geometry proofs

CARL BERGMANN, *Grove School, Madison, Connecticut.*

While "ordered triangles" are a bit unusual, they do provide a fresh way to look at old ideas.

A TYPICAL PROOF in high school geometry is likely to contain something like the following sequence of statements and reasons:

Statement	*Reason*
1. $AC = XZ$	. . .
2. $\angle CAB = \angle ZXY$	. . .
3. $AB = XY$	. . .
4. $\triangle ABC \cong \triangle XYZ$	sas = sas
5. $BC = YZ$	Corresponding parts of congruent figures are equal.



It seems to this writer that such a proof is incomplete inasmuch as Line 4 is insufficient as evidence for Line 5. To make Line 4 adequate we have to add at least the statement that BC and YZ *are* corresponding parts. Fortunately, this fact follows from the same evidence that implies the congruence of the two triangles. Indeed, no instance comes to mind where two figures are proven congruent without the correspondence of their respective parts being simultaneously established.

It would, therefore, be useful to have a compact way of saying that $\triangle ABC$ is congruent to $\triangle XYZ$ with A corresponding to X, B to Y, and C to Z. Let us tentatively express this by saying, "$\triangle ABC$ is correspondingly congruent to $\triangle XYZ$."

We must now examine statements which express this new kind of relation between triangles. Suppose that $\triangle ABC$ is scalene and that $\triangle XYZ$ is such that "$\triangle ABC$ is correspondingly congruent to $\triangle XYZ$" is a true statement. Then the statement "$\triangle BCA$ is correspondingly congruent to $\triangle XYZ$" is false. $\triangle ABC$ and $\triangle BCA$, therefore, are not identical. So, it follows that a new concept has been introduced to replace, or rather supplement, that of the triangle. We may call it "ordered triangle" and write "$\overrightarrow{\triangle ABC}$," "$\overrightarrow{\triangle BCA}$," etc.

The ordered triangle may be defined thus:

Definition I: If A, B, and C are three distinct points, then $\overrightarrow{\triangle ABC}$ is the triplet of directed line segments, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, taken in that order.

More generally we have:

Definition II: If A, B, $\cdots$, Y, Z are distinct points, then

$$\overrightarrow{\text{Polygon } AB \cdots YZ}$$

is the set of directed line segments, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\cdots$, $\overrightarrow{YZ}$, $\overrightarrow{ZA}$, taken in that order.

The relation between conventional triangles and ordered triangles is that the conventional triangle, $\triangle ABC$, may be considered as the class of the six ordered triangles, $\overrightarrow{\triangle ABC}$, $\overrightarrow{\triangle BCA}$, $\overrightarrow{\triangle CAB}$,

$\overrightarrow{\triangle CBA}$, $\overrightarrow{\triangle BAC}$, $\overrightarrow{\triangle ACB}$. Similarly, each ordinary polygon of order n may be considered as a class of $2n$ ordered polygons.

We may now define congruence with respect to ordered polygons thus:

Definition III: Two ordered polygons,

$$\overrightarrow{\text{Polygon } ABC \cdots}$$

and

$$\overrightarrow{\text{Polygon } XYZ \cdots}$$

are congruent if and only if $AB=XY$, $\angle ABC=\angle XYZ$, $BC=YZ$, etc.

Definition III, of course, defines what we originally called "corresponding congruence." No harm is done by dropping the adjective now. Obviously, if we ascribe congruence to a pair of "ordinary" (that is, un-ordered) polygons, it can only be "ordinary" congruence. If we ascribe congruence to a pair of ordered polygons, it follows from Definition III that what we are talking about is what had provisionally been called "corresponding congruence." No confusion results.

The introduction of the concept of the ordered triangle involves a rewording of the theorems in congruence. For instance, "Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other" becomes:

Theorem I: Given $AB=DE$, $AC=DF$, $\angle BAC=\angle EDF$, then $\overrightarrow{\triangle ABC}\cong\overrightarrow{\triangle DEF}$.

The rewording of the other congruence theorems and of the similarity theorems presents no difficulties.

The following new theorems will be needed:

Theorem N1: If

$$\overrightarrow{\text{Polygon } A_1A_2 \cdots A_n}$$

is congruent to

$$\overrightarrow{\text{Polygon } B_1B_2 \cdots B_n,}$$

then for every r such that r is a positive integer less than n,

$$\overrightarrow{\text{Polygon } A_{r+1}A_{r+2} \cdots A_nA_1 \cdots A_r}$$

is congruent to

$$\overrightarrow{\text{Polygon } B_{r+1}B_{r+2} \cdots B_nB_1 \cdots B_r.}$$

Theorem $N1$ follows from Definition II along with the commutative and associative laws for logical multiplication of propositions (that is, along with the fact that the statements that $A_1A_2=B_1B_2$, that $\angle A_1A_2A_3=\angle B_1B_2B_3$, etc., may be rearranged at will) and along with the reversibility of definitions.

Theorem N2: If

$$\overrightarrow{\text{Polygon } A_1A_2 \cdots A_n}$$

is similar to

$$\overrightarrow{\text{Polygon } B_1B_2 \cdots B_n,}$$

then for every r such that r is a positive integer less than n,

$$\overrightarrow{\text{Polygon } A_{r+1}A_{r+2} \cdots A_nA_1 \cdots A_r}$$

is similar to

$$\overrightarrow{\text{Polygon } B_{r+1}B_{r+2} \cdots B_nB_1 \cdots B_r.}$$

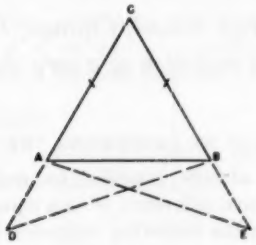
We may also add the following new theorems:

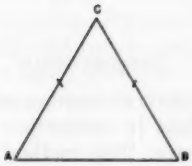
Theorem N3: If $\overrightarrow{\triangle ABC}$ is congruent (or similar) to $\overrightarrow{\triangle BCA}$, then it is equilateral.

Theorem $N3$ is easily proven for congruence by the definitions of congruent triangles and of equilateral triangles. To prove it for similarity, we shall first have to establish that the ratio of similitude is one. Similar theorems may be added with respect to higher order polygons. r and n must be relatively prime in these.

Theorem N4: If $\overrightarrow{\triangle ABC}$ is congruent (or similar) to $\overrightarrow{\triangle CBA}$, $\overrightarrow{\triangle BAC}$, or $\overrightarrow{\triangle ACB}$, it is isosceles with B, C, and A respectively the vertexes.

Given: Isosceles $\triangle ABC$ with $AC = BC$.
To prove: $\angle ABC = \angle BAC$.





Usual Proof

Statement	*Reason*
1. Extend CA to D & CB to E so that $AD = BE$.	A line can be extended indefinitely.
2. $AC = BC$	Given
3. $\angle C = \angle C$	Identity
4. $DA = EB$	Construction
5. $DC = EC$	='s added to ='s give ='s.
6. Draw line segments DB & EA.	Through any two points a line may be drawn.
7. $\triangle DCB \cong \triangle ECA$	sas = sas
8. $\angle BDC = \angle AEC$	cpcf are =
9. $BD = AE$	cpcf are =
10. $\triangle ABD \cong \triangle BAE$	sas = sas
11. $\angle BAD = \angle ABE$	cpcf are =
12. $\angle CAB = \angle CBA$	supplements of equal angles are =

Ordered Triangle Proof

Statement	*Reason*
1. $AC = BC$	Given
2. $\angle C = \angle C$	Identity
3. $\triangle \overrightarrow{ACB} \cong \triangle \overrightarrow{BCA}$	sas = sas
4. $\angle CAB = \angle CBA$	cpcf are =

We are finally ready to apply the concept of the ordered triangle to the proof of a well-known theorem. We shall compare this proof with the one known as the Pons Asinorum, to which it is equivalent. It was, incidentally, a consideration of the Pons Asinorum that triggered the idea of corresponding congruence.

Perhaps the original builder of the Pons Asinorum had this proof in mind, but, having no way of saying that $\triangle ABC$ is congruent to itself in more than one way, he chose the very ingenious detour via the auxiliary triangles DBC and EAC. Since these triangles are distinct, they could be said to be congruent to each other.

Factoring the quadratic trinomial

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An interesting variation of a very old topic.

Introduction

Factoring the trinomial ax^2+bx+c is usually taught in elementary algebra by trial and error. This method is effective whenever the coefficients a and c are relatively small or have few factors. This method, however, becomes cumbersome whenever the number of factors of a and c is large. Consider, for example, the trinomial $64x^2-12x-45$. The number of possible trials is equal to the number of factor pairs of 64 times the number of factor pairs of -45. The total number of possible trials in this example would equal 30. The purpose of this paper is to suggest several other methods of factoring the quadratic trinomial that largely eliminate trial and error.

Factoring by inspection

The method of factoring by trial and error is quite simple when the coefficient of x^2 is unity, for then we need only find factors of c whose algebraic sum is b. It is not difficult, however, to reduce the general case to this particular one. By multiplying and dividing ax^2+bx+c by a (or by a suitable constant less than a), we have

$$ax^2+bx+c=\frac{1}{a}\left[(ax)^2+b(ax)+ac\right]$$

and proceed to factor the expression in brackets with respect to (ax).

Example 1.

Factor $18x^2+21x+5$. We have

$$\begin{aligned}18x^2+21x+5&=\tfrac{1}{2}\left[(6x)^2+7(6x)+10\right]\\&=\tfrac{1}{2}(6x+5)(6x+2)\\&=(6x+5)(3x+1).\end{aligned}$$

Factoring by completing the square

It is always possible to reduce ax^2+bx+c to a difference of two squares as is shown in the following sequence of steps:

$$\begin{aligned}&ax^2+bx+c\\&=a\left[x^2+\frac{b}{a}x+\frac{c}{a}\right]\\&=a\left[\left(x^2+\frac{b}{a}x+\frac{b^2}{4a^2}\right)-\frac{b^2-4ac}{4a^2}\right]\\&=a\left(x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right)\left(x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right).\end{aligned}$$

Although the last expression could be used as a formula, it is recommended that the student work through the process in each particular problem.

Example 2.

Factor $5x^2+14x-3$. We have

$$\begin{aligned}&5x^2+14x-3\\&=5\left(x^2+\frac{14}{5}x-\frac{3}{5}\right)\\&=5\left[\left(x^2+\frac{14}{5}x+\frac{49}{25}\right)-\frac{64}{25}\right]\\&=5\left(x+\frac{7}{5}+\frac{8}{5}\right)\left(x+\frac{7}{5}-\frac{8}{5}\right)\\&=(x+3)(5x-1).\end{aligned}$$

The method of undetermined coefficients

Assume that ax^2+bx+c has the factors $px+q$ and $rx+s$. Since $ax^2+bx+c=$

$(px+q)(rx+s)$ is an identity, the equality must hold for all values of x. Choosing four values for x, we can determine the values of p, q, r, s uniquely. The computation will be further facilitated if for a suitable choice of x, ax^2+bx+c is prime, as is illustrated in the next example.

Example 3.

Let us test this method in the example given in the introduction. Let

$$64x^2-12x-45=(px+q)(rx+s).$$

Setting $x=1$ and $x=-1$ in turn, we get

$$(p+q)(r+s)=7 \text{ and } (p-q)(r-s)=31.$$

We may choose $p+q=7, p-q=1, r+s=1, r-s=31$; so that $p=4$, $q=3$, $r=16$, $s=-15$; and $64x^2-12x-45=(4x+3)(16x-15)$.

Factoring by the factor theorem

Suppose $x=r$ is a zero of ax^2+bx+c, that is, a root of $ax^2+bx+c=0$. To obtain such values of x, we need not necessarily solve the equation. Frequently, mere inspection will yield such a number. Then by the factor theorem, $(x-r)$ is a factor, while the other factor can be obtained by division.

Example 4.

Factor $(3x^2+7x-6)$. By inspection $x=-3$ is a zero. Hence $(x+3)$ is a factor and, by division, $(3x-2)$ is the other factor.

Administrative and guidance personnel can assist the classroom teacher to provide for rapid and slow learners in ways already described; but an equal, if not greater, role remains for the teacher. It is the teacher who adapts methods, uses materials, and provides experiences suitable to the individual requirements of his rapid and slow learners. He, often with the help of his students, is the one who provides enrichment opportunities, organizes small groups and committees, suggests individual projects, motivates continuous developmental growth, encourages individual research and creative activity, suggests reading materials within the comprehension ability of pupils, and leads boys and girls to evaluate and chart their progress in the direction of significant goals.—*Teaching Rapid and Slow Learners in High Schools the Status of Adaptations in Junior, Senior, and Regular High Schools Enrolling More than 300 Pupils.*—Bulletin 1954, *No. 5, Office of Education, U. S. Department of Health, Education, and Welfare.*

Edited by Phillip S. Jones, University of Michigan, Ann Arbor, Michigan

Mathematics and philosophy[1]

by R. H. Moorman, Tennessee Polytechnic Institute, Cookeville, Tennessee

Introduction

The writer has been studying the historical relationships of mathematics and philosophy for nearly twenty years. Such relationships are recognized as being of more importance in our present atomic age than they were a few years ago. Thus, Thornton Fry has recently declared:[2]

> If we are to train men to meet this new and challenging future, we must train them in those things which are most lasting in our science—its habits of thought, its fundamental disciplines, its peculiar relation to philosophy and the other sciences.

The history of mathematics and the history of philosophy might be thought of as surfaces which intersect in the lives of many men. While it is possible to write an entire book[3] on the meaning of mathematics, it will be sufficient for this paper to define mathematics as the science of relationships. The term philosophy means etymologically the "love of wisdom," but its meaning has changed since the Greeks originated the term. Today, philosophy is the critical and comprehensive study of the ultimate problems that are raised in the search for knowledge; it is the $(n+1)$th science. As Demiashkevich declared:[4]

> Philosophy, then, with regard to the various sciences, has the function of a clearing house. In the light of a clear, critical evaluation, it strives to build a general outlook upon life, to decide what, if anything, has become certain scientifically; to decide also what remains scientifically uncertain, with relation to the knowledge of the first cause and final end of human existence.

In the atomic age we are beginning to realize that we must use philosophy to guide the proper use of atomic energy.

Pythagoras[5]

The first Greek mathematician who was also outstanding in philosophy was Thales, about 600 B.C. He introduced deductive demonstration into geometry, but what we know of his philosophy showed little or no evidence of the influence of mathematics.

Pythagoras, who lived about 525 B.C., is famous as a mathematician and also as

[1] Presented as the chairman's address to the Southeastern Section of the Mathematical Association of America, Athens, Georgia, March 16, 1956.

[2] T. C. Fry, "Mathematics as a profession today in industry," *American Mathematical Monthly*, Vol. 63, p. 79.

[3] R. Courant and H. Robbins, *What Is Mathematics?* Oxford University Press, 1941.

[4] Michael Demiashkevich, *An Introduction to the Philosophy of Education*, American Book Co., 1935, p. 25.

[5] For a fuller discussion, see R. H. Moorman, "Pythagoras: mathematician and philosopher," *The Pentagon*, Vol. 8, pp. 79–84.

a philosopher. In studying cosmology, that is, the problem of the fundamental "stuff" of the universe, earlier philosophers had come to the conclusion that it was *earth*, or *air*, or *fire*, or *water*. Pythagoras rejected these earlier beliefs and decided that the primary constituent of the material world was *number*. In reflecting on the quality that all things must have in common, he decided that all things could be counted. Thus, he gave objective reality to the concept of number and tried to build a universal mathematics based on commensurable numbers. For him all even numbers were feminine and all odd numbers were masculine. Since one was not recognized as a number, four was the first square number and was *justice*. Aristotle criticized Pythagoras for carrying his mathematical deductions and abstractions too far in his consideration of the problems of ethics:[6]

> Pythagoras first attempted to speak about virtue, but not successfully; for by reducing the virtues to numbers he submitted the virtues to a treatment which was not proper to them. For justice is not a square number.

The number five was *marriage* since it was the union of the first masculine number and the first feminine number. To him numbers were everything—even when a man had indigestion it was because the numbers in his stomach were not straightened out. The Pythagoreans built up a number mysticism of which we still see traces today.

In the use of the Pythagorean Theorem a difficulty was encountered since the hypotenuse of a right triangle with unit legs could not be expressed as an integer. The hypotenuse was thus called *irrational*, originally meaning "inexpressible." The Pythagoreans kept secret the fact that they had discovered irrational numbers and the story was told that they drowned one member of the brotherhood for revealing the secret to an outsider. The theory of irrational numbers was not completely worked out until the researches of Cantor and Dedekind in the latter part of the nineteenth century.

Plato[7]

Plato flourished about 375 B.C., and he is more famous as a philosopher than as a mathematician. He introduced into mathematics the idea that constructions should be made by using only unmarked straightedges and compasses. This had philosophical implications since his reason for thus limiting constructions was that mathematics could be kept more pure in this way. This led to the three famous problems of antiquity which were only shown to be impossible in the last half of the nineteenth century. We see reports of circle squarers and angle trisectors quite often in the press today. Efforts to solve the three famous problems of antiquity have led to many mathematical discoveries.

Another significant contribution of Plato to mathematics was the introduction of analysis into geometry. In the analytic approach he sought for sufficient conditions to lead from the conclusion of a theorem back to the hypothesis. This type of analysis was introduced into the solution of problems in philosophy by Descartes. Plato insisted upon accurate definitions, clear assumptions, and logical proof. He laid the foundation for the work of Euclid, who came about a century later. For example, Plato urged the examination of hypotheses in the *Republic*, declaring:[8]

> . . . as to the mathematical arts . . . never can they behold the waking reality so long as they leave the hypotheses which they use undisturbed, and are unable to give an account of them. For when a man knows not his own principle, and when the conclusion and intermediate steps are also constructed out of he knows not what, how can he imagine that such arbitrary agreement can ever become science?

[6] W. D. Ross, *The Works of Aristotle, Magna Moralia*, 1182a.

[7] For fuller discussion, see R. H. Moorman, "The influence of mathematics on the philosophy of Plato," *The Pentagon*, Vol. 12, pp. 51–61.

[8] *The Dialogues of Plato*, translated by B. Jowett, Charles Scribner's Sons, 1908, Vol. 2, p. 361.

Plato had no definite system of philosophy since he did not believe in the necessity of systems of philosophy. His philosophy, however, may be described by the term *idealism*, which is merely a euphonious expression of idea-ism. For him, ultimate realities were ideas. Man was like a person chained in a cave with his back to the opening. All that he could know about the world was learned by studying the flickering shadows reflected on the back of the cave. Plato's concept of ideas was undoubtedly influenced by his study of geometrical forms.

Plato's *universal mathematics* consisted in the belief that the mathematical concepts of *order* and *measure* could be applied to every phase of life. In the *Republic* he declared:[9]

> The arts of measuring and numbering and weighing come to the rescue of the human understanding—that is the beauty of them—and the apparent greater or less, or more or heavier, no longer have the mastery over us, but give way before calculations and measure. . . .

Plato believed that *order* and *measure* were the key to health and to the practice of medicine, stating in the *Republic*:[10]

> And the creation of health is the creation of natural order and government of one another in the parts of the body, and the creation of disease is the production of a state of things at variance with this natural order.

In the *Protagoras*, Plato declared:[11]

> Would the art of measuring be the saving principle, or would the power of appearance? Is not the latter that deceiving art which makes us wander up and down and take the things at one time of which we repent at another, both in our actions and in our choice of things great and small? But the art of measurement is that which would do away with the effect of appearances, and showing the truth, would fain teach the soul at last to find rest in the truth, and would save our life.

While we cannot deal with it in detail here, it seems evident that Plato tried to apply mathematical *order* and *measure* to all of the problems of philosophy with which he dealt.

[9] Plato, *op. cit.*, Vol. 2, p. 433.
[10] Plato, *op. cit.*, Vol. 2, p. 271.
[11] Plato, *op. cit.*, Vol. 1, p. 156.

Aristotle[12]

Aristotle, who reached the peak of his activities about 340 B.C., studied in Plato's Academy. Just as Plato is said to have placed over the door of his Academy, "Let no one ignorant of geometry enter here," so Aristotle assumed a mastery of elementary mathematics on the part of the students in his Lyceum. He framed his system of logic on models from mathematics. Aristotle wrote about a method in mathematics called "verging" that was used in attempts to trisect the angle. In his writings we find many of the concepts of irrationals, the continuum, and the infinite. His discussion of the errors in measurement would do justice to any twentieth century teacher trying to explain the approximate numbers obtained by measurement.

Aristotle is generally regarded as the greatest thinker in Greek life. He was a universal genius in philosophy, and no philosopher ever proposed and defended a philosophy as well as he did. Aristotle opposed Plato's *ideas* and, to replace these ideas, he introduced *forms*. For example, he held that the form of a building in the mind of the architect was in some way the cause of the building, having a tendency to produce a concrete reality like itself. Aristotle did not consider the role of mathematics to extend to philosophy as immediately as many other mathematicians who have philosophized. For him the "things" of mathematics were not real, while the things dealt with by philosophy were real. In the *Politics* he declared:[13]

> The physicist takes account not only of the attributes of physical bodies but of their matter; though he speaks of planes and lines and points, he speaks of these only as limits or extremities of a physical and movable body. The mathematician says nothing of matter, his planes, lines, and points are the result of ab-

[12] For a fuller discussion see R. H. Moorman, "The influence of mathematics on the philosophy of Aristotle," *The Pentagon*, Vol. 16, pp. 17–23.
[13] Thomas Heath, *Mathematics in Aristotle*, Clarendon Press, 1949, p. 13.

straction and are considered apart altogether from physical bodies.

Aristotle, like other mathematicians who philosophize, had the concept of a *universal mathematics.* With most mathematicians this has been the theoretical application of mathematics to every phase of life, but Aristotle seems to have been more cautious in his extension of mathematics.

Aristotle recognized the importance of mathematics in having a proper method of arriving at the truth, for he declared that deliberation and calculation are identical, thus implying that all thought is mathematical thought.

Aristotle invented the syllogism as a method of reaching conclusions by means of deductive reasoning. This became the method of proof used in geometry and it was used extensively in philosophy from that time. His Law of the Excluded Middle, that is, that a thing must either be or not be, became the method of reductio ad absurdum. This was used as the basis of indirect proofs in geometry and had wide application in philosophical problems. His Law of Contradiction, that a thing cannot at the same time both be and not be, found wide application in his philosophy. Aristotle maintained that the planets moved in circular orbits because the circle is the most perfect curve. This was, of course, an error, for there is nothing to show that perfection is a principle of control in astronomy.

Plato had held that there should be a sort of "moral calculus" in dealing with ethics, declaring that there should be an effort to reduce the sum of our human sins to a minimum. Aristotle said that virtue is the *golden mean* between two opposite vices. In the *Nicomachean Ethics* he declared:[14]

> It appears then that virtue is a kind of mean because it aims at the mean.
>
> On the other hand, there are many different ways of going wrong; for evil is in its nature infinite, to use the Pythagorean phrase, but good is finite and there is only one possible way of going right. So the former is easy and the latter is difficult; it is easy to miss the mark but hard to hit it and so by our reasoning excess and deficiency are characteristic of vice and the mean is a characteristic of virtue.

Aristotle was the first philosophical writer to make a strict separation of the branches of philosophy, and mathematics seems to have been related to his treatment of all the major problems of philosophy. The influence of mathematics on his philosophy was not as great as in the case of other mathematical philosophers, because he thought the things dealt with by mathematics were not real, while the things dealt with by philosophy were real.

The Middle Ages

During the Middle Ages both mathematics and philosophy went into a decline in Europe. Had it not been for the Arabs probably all the mathematical knowledge of the Greeks would have been lost. Western civilization did not get back to the level of attainment of the Greeks in either mathematics or philosophy until the sixteenth century.

The dominant philosophies of the Middle Ages were realism and nominalism. Realism was the mediaeval interpretation of Plato's philosophy, holding that there were certain universal "reals," correresponding to Plato's ideas. The realists held, for example, that there was a "real" circle, more perfect than any circle that existed anywhere in the universe. Nominalism was the mediaeval interpretation of Aristotle's philosophy. The nominalists declared that there were no "real" circles, but that the universal reality consisted only in the name "circle," thus the term nominalism.

Aristotle's views on science were adopted by the church in the Middle Ages and it became irreligious to disagree with any of his conclusions. Throughout the Middle Ages, Aristotle was the one great

[14] J. E. C. Welldon, *Aristotle—on Man in the Universe,* Classics Club, 1943, p. 108.

authority on science and philosophy. Until late in the eleventh century, under the predominant influence of Platonism or Neo-Platonism, realism was the usual view. Roscelinius, about 1100, was the first nominalist, that is to say, he denied the independent reality of universals and regarded general terms as mere names. Abelard (about 1125) had the viewpoint that universals were embodied in individual things. He thought that "forms" or "ideas" were conceived by God from all eternity. The human mind could only get to know them by abstraction from particulars. Aristotelian Scholasticism culminated in Thomas Aquinas who flourished about 1250. Like Aristotle, Aquinas laid great stress on the distinction between matter and form, though he was rather liberal in his conception of the numerous "forms" which could exist apart from matter.

Demiashkevich declared that the realists were influenced by geometry while the nominalists were influenced by physics. Since mathematics influenced the philosophies of both Plato and Aristotle, it seems likely that it had an influence on both realism and nominalism.

In commenting on the philosophy of the Middle Ages, De Wulf declared:[15]

> Speculative or theoretical . . . philosophy gives the results of acquaintance with the world in its *objective* aspect; it includes the philosophy of nature, mathematics and metaphysics, which consider . . . change, quantity and the general conditions of being, respectively in the material world. There are three stages through which the mind passes in order to secure a total view of the world of which it is spectator. The Middle Ages defines physics, or the philosophy of nature, as the "study of the material world. . . ." *Mathematics*, which studies quantity as regards its logical implications, was for the ancients a philosophical and therefore a general science, and in our day many scientists are tending to return to this Aristotelian notion. *Metaphysics* enters deepest of all into reality and deals with what is beyond motion and quantity. . . .

The quadrivium of liberal arts in the Middle Ages consisted of arithmetic, astronomy, geometry, and music. Demiashkevich declared that geometry's place in the quadrivium was significant in the philosophy of the Middle Ages.

Descartes[16]

Descartes, who lived from 1596 to 1650, is considered the father of modern mathematics and of modern philosophy. Although he is the prime inventor of analytic geometry, he placed his only systematic treatment of geometry as an appendix to his discussion of philosophical method, *Discours de la Méthode*, published in 1637. He made many contributions to the theory of equations but these, too, seemed incidental to his discussion of philosophy. Of the many men who have been famous as mathematicians and as philosophers, Descartes was perhaps the most outstanding.

At the time of Descartes the term "philosophy" had a much wider meaning than it has today. Philosophy included all there was of what we call "science," while the term "science" simply connoted any branch of knowledge. Descartes made no distinction whatever between science and philosophy, declaring:[17]

> . . . the word Philosophy signifies the study of wisdom, and by wisdom is to be understood not merely prudence in the management of affairs, but a perfect knowledge of all that man can know, as well for the conduct of his life as for the preservation of his health and the invention of all the arts.

Descartes made it clear that he considered his work in optics and meteorology to be parts of philosophy. Thus, he declared in his letter to Dinet:[18]

> In this small number of meditations which I published, all the principles of the philosophy which I am preparing are contained; and in the Dioptric and Meteors I have deduced from these principles many particular things which show

[15] Maurice De Wulf, *Philosophy and Civilisation in the Middle Ages*, Dover, 1953, pp. 91–92.

[16] For a fuller discussion, see R. H. Moorman, "The influence of mathematics on the philosophy of Descartes," *National Mathematics Magazine*, Vol. 17, pp. 296–307.

[17] René Descartes, *Les Principes de la Philosophie*, Henry Le Gras et Edmé Pépinque, 1651, Preface.

[18] E. S. Haldane and G. R. T. Ross, *The Philosophical Works of Descartes Rendered into English*, Cambridge University Press, 1912, Vol. 2, p. 375.

what is the manner of my reasoning; and that is why, although I am not yet setting forth all that philosophy, I yet consider that what I have already given forth suffices to make known what it will be.

The study of the influence of mathematics on the philosophy of Descartes includes the study of the influence of mathematics on the science with which he dealt. At an earlier period there was no science as we understand the meaning of the term today. It was precisely at the time of Descartes that modern science began to develop. Thus, to show the influence of mathematics on science after the time of Descartes would not show its influence on philosophy.

The concept of a universal mathematics was prominent in the philosophy of Descartes, and he designated it by the term *mathesis*. Mathesis included the entire scope of *order* and *measure*, of which ordinary mathematics was the husk rather than the core, the envelope rather than the contents. Examination of the structure of ordinary algebra suggested to Descartes the idea of constructing a universal characteristic of which the former would be only a restricted application.

Descartes invented analytic geometry and in philosophy he considered analysis to be a more important technique than synthesis. But in philosophy his analysis was not the analysis of his mathematics; rather it was the analysis which had been invented by Plato and used by the Greek geometers. Descartes promised to extend the solution by mathematical analysis to problems of any type whatever, but no tangible evidence of such efforts has ever been found. Only in the twentieth century has symbolic logic extended mathematical analysis to all types of propositions. This has been the work of Whitehead, Russell, and others. One might be inclined to think that the same techniques are used in philosophy, but only much later than they were used in mathematics.

In epistemology Descartes' universal mathematics was the theoretical extension of order and measure to any type of knowledge whatever. Mathematics appeared to be the only hope of finding an entering wedge to the problem of getting exact knowledge. Thus, Descartes declared in the *Discourse on Method*:[19]

> I was especially pleased with mathematics on account of the certitude and evidence of its reasons; but I did not notice at all its true usage, and considering only the mechanical arts, I was surprised that foundations so strong and solid had not been used for larger buildings.

His procedure was to admit no idea which was not as clear and distinct as the foundations of mathematics. Descartes sought to find by means of analysis a clear and distinct idea to use as the foundation of knowledge. Just as a mathematician analyzes a problem to find a datum to which are related all the variables of the problem, Descartes obtained the principle expressed in the phase *cogito ergo sum* as the product of his analysis. The *cogito* was an axiomatic statement which might be thought of as the fulcrum of Descartes' system of philosophy.

In reply to certain of his critics, Descartes reluctantly agreed to display some of his metaphysics in the synthetic geometrical form of Euclid's *Elements*. The result was Descartes' "Reasons Which Prove the Existence of God and the Difference between the Mind and Body of Man, Displayed in Geometrical Fashion." There were ten definitions, seven postulates, ten axioms, four propositions, and one corollary. Some men concluded that the deductive synthetic expression of Descartes' metaphysics was the significant use of mathematics in his philosophy. This became the almost worthless *more geometrico* of the eighteenth century. Some writers, seeing that the *more geometrico* was of little value, have discounted the importance of mathematics in the philosophy of Descartes. They have overlooked the fact that he used analysis, similar to mathematical analysis, as the essential technique in forming his metaphysics.

[19] René Descartes, *Discours de la Méthode*, Leyden: Jan Marie, 1637, p. 9.

In dealing with the problem of natural philosophy, Descartes gave objective reality to *geometrical extension* in the same way that Pythagoras gave objective reality to number. Descartes regarded all material bodies as being in reality mere modifications of geometrical extension. His universal mathematics attained its highest development in natural philosophy. Order and measure were extended theoretically to all the problems of mechanics, music, physics, astronomy, biology, and optics. Such a concept was an innovation in the seventeenth century.

Descartes attempted to give a complete mathematical and mechanical interpretation of the universe, which was alien to the whole temper of the Middle Ages and to the Aristotelianism of his own age. Before the time of Descartes the origin and maintenance of the physical universe had been explained by means of "souls." Descartes advanced the concept that the universe was derived and maintained by general laws. His theory of vortices had no validity, but it anticipated the more accurate theory of Newton. The theory of universal gravitation was far more accurate and mathematical than the theory of vortices. Newton discredited Descartes and has in turn succumbed to Einstein, and who can say when Einstein's theory will be superseded by another?

Spinoza[20]

Spinoza, who lived from 1632 to 1677, has been considered as being everything ranging from an atheist to a "God-intoxicated Jew." His first work to be published was a geometrical treatment of Descartes' philosophy (*R. des Cartes Principiorum Philosophiae, More Geometrico Demonstrata,* 1663). His most important work was his *Ethics* (*Ethica More Geometrico Demonstrata,* 1677), which was not published during his lifetime because his contemporaries considered it atheistic. Spinoza was not an outstanding mathematician but his name is included in the *Scripta Mathematica* list of ten "philosophers who were also mathematicians."

In philosophy, Spinoza was one of the group of leading continental rationalists. Spinoza was a pantheist, taking the two ultimate realities of Descartes' system of philosophy (mind and matter or thought and extension) and uniting them into a single reality or substance, namely, God.

Bertrand Russell declared that the influence of mathematics on the philosophy of Spinoza was very obvious. In general, his point of view was similar to that of Descartes, but there are some significant differences. Mathematics was clearly related to the rationalistic method of Spinoza.

Spinoza used the synthetic geometrical form of Euclidean geometry in his treatment of Descartes' philosophy, in the appendix to the *Short Treatise,* and in his major work, the *Ethics.* In the geometrical treatment of Descartes' philosophy there were 23 definitions, 37 axioms, and 61 propositions. The *Ethics* contained 26 definitions, 15 axioms, 2 postulates, and 259 propositions. Some of Spinoza's predecessors had made casual attempts to use this form, but he used it consistently.

Descartes seemed to think that the problems of practical philosophy, ethics, esthetics, and politics could not be treated with mathematical exactitude and, therefore, should not be dealt with at all. Spinoza, on the other hand, was primarily interested in ethics and tried to deal with ethics in a mathematical way. Thus, in denying human freedom, he declared:[21]

> For the present I wish to revert to those, who would rather abuse or deride human emotions than to understand them. Such persons will doubtless think it strange that I should attempt to treat of human vice and folly geometrically, and should wish to set forth with rigid reasoning those matters which they cry out against as repugnant to reason, frivolous, ab-

[20] For a fuller discussion, see R. H. Moorman, "The influence of mathematics on the philosophy of Spinoza," *National Mathematics Magazine,* Vol. 18, pp. 198–215.

[21] R. H. M. Elwes (translator), *Philosophy of Benedict Spinoza,* Tudor Pub. Co., 1936, p. 128.

surd, and dreadful. However, such is my plan . . . I shall, therefore, treat of the nature and strength of the emotions according to the same method, as I employed heretofore in my investigations concerning God and the mind. I consider human actions and desires in exactly the same manner, as though I were concerned with lines, planes, and solids.

Spinoza's theory of political philosophy was also influenced by mathematics. His estimate of the influence of mathematics on his philosophy in general was given in a letter he wrote near the end of his life:[22]

. . . you ask . . . me, "How I know that my philosophy is the best among all that have ever been taught in the world, or are being taught, or ever will be taught?" a question which I might with much better right ask you; for I do not presume that I have found the best philosophy, I know that I understand the true philosophy. If you ask in what way I know it, I answer: In the same way as you know that the three angles of a triangle are equal to two right angles; that this be sufficient, will be denied by no one whose brain is sound. . . .

Leibniz[23]

Leibniz (1646–1716) was the third of the great continental rationalists in philosophy and was one of the greatest of modern mathematicians. While Leibniz published a number of well-known works, his less well-known *Dissertatio de Arte Combinatoria* (1666) is probably the most important in the consideration of the influence of mathematics on his philosophy.

Through his invention of the calculus, Leibniz is considered a great mathematician, but he published no book in mathematics, only a few fragments. Leibniz worked problems in mathematics throughout his life, and his knowledge of mathematics can be seen in his correspondence. There was a great controversy as to whether he got his ideas of the calculus from Newton. Authorities now generally agree that Leibniz invented the calculus independently of any knowledge of Newton's fluxions, though Newton had the idea of calculus earlier than Leibniz. Leibniz was a genius in mathematical notations, and his calculus notation is still used. He invented a calculating machine that would add, subtract, multiply, divide, and take roots. He invented determinants without any knowledge of the prior Japanese work, laid the foundation for the theory of envelopes, and introduced the terms *axes* and *coordinates*.

In philosophy Leibniz was of the opinion that he should first see how near his predecessors had come to the truth, in contrast with Descartes, who thought that his first duty was to doubt all the conclusions of his predecessors. Leibniz harmonized the separation of *mind* and *matter* by Descartes, by using what he called *pre-established* harmony. He declared that mind and matter were like two clocks set to run together, being synchronized by God.

The most important tenet in the philosophy of Leibniz was his doctrine of *monads*. For him, substance was essentially active rather than passive, or in other words, "To be is to act." The centers of activity were the monads. The doctrine of monads bore a slight similarity to modern quantum theory. Another important tenet in Leibniz' philosophy was his law of continuity, which declared that there was neither vacuum nor break in nature, but that the different species of creatures arose by imperceptible degrees from the lowest to the most perfect form.

The most significant effect of mathematics on Leibniz' philosophical method was his *universal mathematics*. He carried out this idea more fully than Descartes had done and tried to interest Louis XIV of France in the use of a system of universal symbols. In his early years Liebniz declared that simple and primitive concepts should be symbolized by prime numbers and that the combination of two concepts should be represented by their product. Thus, if three represented "rational" and seven represented "animal," then twenty-one would represent "man." He used negative numbers also, but the

[22] Elwes, *op. cit.*, p. 182.

[23] For a fuller discussion, see R. H. Moorman, "The influence of mathematics on the philosophy of Leibniz," *National Mathematics Magazine*, Vol. 19, pp. 131–140.

scheme broke down and he gave up the system of representing concepts by number. Throughout his life Leibniz sought a universal mathematics which was to be expressed in terms of a *universal characteristic*. Although he was never able to work out this universal characteristic, Leibniz was important in the historical evolution of symbolic logic, declaring that he tried to reduce logic to the same state of certainty as arithmetic. In his *New Essays on the Human Understanding*, Leibniz declared:[24]

> I begin to form for myself a wholly different idea of logic from that which I formerly had. Then I regarded it as a scholar's diversion, but now I see that . . . it is like a universal mathematics. . . .
>
> There are very many examples of demonstrations outside of mathematics and one can say that Aristotle gave some of them in his first *Analytiques*. Indeed logic is just as susceptible to demonstration as geometry; and one can say that the logic of the geometers is based on the work of Euclid or a particular extension of general logic.

According to Lewis and Langford:[25]

> The program both for symbolic logic and for logistic, in anything like a clear form, was first sketched by Leibniz, though the ideal of logistic seems to have been present as far back as Plato's *Republic* . . . Leibniz correctly foresaw the general character which logistic was to have . . . he failed of any clear understanding of the difficulties to be met, and he contributed comparatively little to the successful working out of details. Leibniz expected that the whole of science would shortly be reformed by the application of this method. This was a task clearly beyond the powers of any one man. . . .

Bertrand Russell is one of the scholars who has brought symbolic logic to its maturity. This is what Russell had to say about Leibniz' part in the development of symbolic logic:[26]

> He seems to have thought that the symbolic method in which formal rules obviate the necessity of thinking, could produce everywhere the same fruitful results as it has produced in the sciences of number and quantity. . . . "If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to set down to their stalls, and say to each other (with a friend as witness, if they liked): 'Let us calculate.'"

Leibniz' philosophical thought was influenced by his work in the method of the calculus. In dealing with natural philosophy his theory was that the changes in nature were infinitesimally small. This theory was rather closely related to his concept of the infinitesimal calculus. Just as the volume of a solid could be found by integration, so could the entire external world be considered as an aggregate of infinitesimals.

Utilitarian Sciences

While it might be possible to define a science as any organized body of knowledge, we need to distinguish between utilitarian sciences and non-utilitarian sciences. A *utilitarian* science may be defined as one that is of practical benefit to mankind, as contrasted with the earlier *non-utilitarian* stage of a science in which there is merely description and classification. Utilitarian sciences arise from the mathematization of philosophy. Astronomy began to develop from astrology about 130 B.C. when Hipparchus invented trigonometry to put it on a mathematical basis. Physics developed from the mathematization of natural philosophy at the time of Descartes by the union of quantitative or mathematical relationships with observation and experimentation. Chemistry grew out of alchemy by the mathematization of philosophy. Boyle began a new era in chemistry about 1660 with his quantitative law. In the nineteenth century biology began to be a utilitarian science with the quantitative work of Mendel.

The mathematization of philosophy means more than the collection of statistics or the making of studies. It must be possible to prove functional relationships.

[24] J. M. Bruyset (editor), *Esprit de Leibniz, ou Recueil de Pensées Choisies*, Lyon: Bruyset, 1772, Vol. II, p. 137.

[25] C. I. Lewis and C. H. Langford, *Symbolic Logic*, Century, 1932, p. 4.

[26] Bertrand Russell, *A Critical Exposition of the Philosophy of Leibniz*, London: Allen and Unwin, 1937, p. 169.

Today we have many branches of knowledge such as psychology, sociology, and education that are trying to be very scientific. But people in these fields of knowledge are drawing conclusions that a mathematically-trained person would never draw. It is said that an astronomer can predict where every star will be at midnight but he cannot predict where his daughter will be. There are many pseudosciences that claim validity which is not justified. The public, untrained in mathematics and philosophy, is accepting their conclusions. It is possible to collect statistics on any subject, even sex, but that does not give validity to such a collection as the Kinsey report. The general public needs to understand the fundamental concepts of mathematics and of philosophy. Too much of our education today shows only the applications or final results rather than teaching people how to judge the validity of the results. Unless people are taught to doubt conclusions until they understand the theoretical background, we will always find the public taken in by such pseudoscientific writing as *The Search for Bridey Murphy*. The government has been blamed for suppressing scientific information, but the fault lies with our system of education which tries to get to the conclusions without an adequate theoretical background in mathematics and philosophy.

"First, therefore, amongst so many great foundations of colleges in Europe I find it strange that they are dedicated to professions, and none left free to arts and sciences at large. For if men judge that learning should be referred to action, they judge well, but in this they fall into the error described in the ancient fable, in which the other parts of the body did suppose the stomach had been idle, because it neither performed the office of motion as the limbs do, nor of sense, as the head doth, but yet notwithstanding it is the stomach that digesteth and distributeth to all the rest. So if any man think philosophy and universality to be idle studies, he doth not consider that all professions are from thence served and supplied. And this I take to be a great cause that hath hindered the progression of learning, because these fundamental knowledges have been studied but in passage. For if you will have a tree bear more fruit than it used to do, it is not anything you can do to the boughs, but it is the stirring of the earth and the putting new mould about the roots that must work it."—Francis Bacon.—*Taken from "Mathematics and the Future of Science," by Marshall H. Stone,* Bulletin of the American Mathematical Society, *Volume 63, Number 2, March 1957.*

• MATHEMATICS IN THE JUNIOR HIGH SCHOOL

Edited by Lucien B. Kinney, Stanford University, and Dan T. Dawson, Stanford University, Stanford, California

Teaching percentage for understanding and use

by Lucien B. Kinney

That ability to use and interpret per cent is expected of every member in our society is evident as we read newspapers and magazines, listen to and watch television, or talk with others. We learn that a bank offers a 3 per cent interest rate on savings; that one food has 25 per cent less calories than another; that 50 per cent of families in the country have incomes over $4800; that the price index rose ½ per cent last month, while department store sales rose 5 per cent. For mail orders add 4 per cent tax if you live in this locality, and 3 per cent otherwise.

Per cent is an important part of adolescent as well as adult language. While confused by his classroom study of per cent, the pupil is taking it for granted in games won and lost, batting averages, pitching and fielding records, and other athletic statistics. Illustrations at both levels could be elaborated indefinitely; per cent has pervaded our language and anyone who does not understand it is only partially literate.

In recognition of its wide usage, the curriculum in grades 7 and 8, and in ninth grade general mathematics, has long given considerable attention to per cent. Yet the reputation for difficulty that per cent has attained, both among teachers and pupils in the junior high school, and also the incidence of incorrect usage are indications that the results are not wholly successful. Thus we encounter advertisements of "prices reduced 100 per cent," or reports that the population has increased 200 per cent, when the writer intended to say that the population had doubled. The reader can add other illustrations from his own experience.

Difficulties in teaching per cent

Some of this reputation for difficulty arises from the very facts that make per cent so useful in our language; namely, that it offers a concise and simple way of expressing comparisons. Many of the relationships in which per cents are used are implied rather than explicit. Thus, when we say "the population has increased 40 per cent over the past 10 years," we do not need to say 40 per cent *of what*. Usages in per cent are based on conventions that require increases to be calculated upon the original measurement. This and other conventions need to be learned explicitly.

Another difficulty arises from the fact that percentage is typically taught in a vacuum. The percentage calculations, for example, "Cases I, II, and III," are frequently thought of as unique to per cent, and studied without reference to the same calculations with decimal and common fractions. The applications are to

business situations rather than those familiar to pupils, who remain unaware of the extensive use of per cent in their daily affairs.

These difficulties are only symptoms of a failure to present per cent meaningfully. Through grade six, the need for meaningful learning of arithmetic has been clearly recognized in modern texts and syllabi. Emphasis is placed on such understandings as these:

The concepts basic to the operation as indicated by correct usage of vocabulary.

The rationale of the operation—why it works.

The relationship of this operation to other operations.

The typical situations that call for this operation in daily life, and how they may be recognized.

These understandings are developed inductively in the initial setting in which the operation is presented and in the variety of concrete situations which lead to eventual manipulation at the abstract level.

When these same syllabi and texts reach percentage, however, this approach is usually abandoned. Given the definition "per cent means hundredths," the pupil is expected to acquire the basic concepts and relationships with little further experience. The calculations with per cent appear as something new and different.

Meaningful teaching of per cent

A meaningful approach in the teaching of per cent must recognize these facts:

1. Per cent is a language widely used because of its convenience for expressing ratios. These same ratios may also be expressed with common and decimal fractions.
2. There are no new computations to learn (unless translating a decimal fraction to per cent is considered as such).
3. The new requirement is that of learning to express and interpret, in the language of per cent, the relationships in situations commonly encountered both in adult and adolescent affairs.

These ideas can be emphasized if per cent is introduced and followed up at four levels which, for convenience in description, we may refer to here as "topics." The first two topics are directed primarily to the language aspects of per cent. The third and fourth topics deal with the use of per cent in problem-solving. These topics may be summarized and illustrated.

Topic I. Using division to compare quantities

When division is used in comparisons, the ratio may be expressed as a common fraction in which the numerator is being compared to the denominator. If the division is carried out, the dividend is being compared to the divisor, and ratio is the quotient, a decimal fraction. Later on in the topic this is expressed as per cent. The situations in which the comparisons are made are those familiar to the pupils, including units of measure, athletic records, and ratios arising in other activities. The major attention is on ratio, with per cent only one of three ways of expressing ratio. Of the major percentage calculations, only Case II—finding what per cent one number is of another—is taught in this topic. The collection of applications brought in to post on the bulletin board and the situations discussed in class are designed to develop an understanding of ratio and an appreciation of the usefulness of per cent.

Topic II. Expressing ideas with graphs

The purpose here is to continue the emphasis on the language aspects of per cent and extend the manipulations to include Case I—finding a per cent of a number. This operation is necessary in the construction of divided bar graphs and circle graphs.

While the primary purpose of the topic is to develop skill in constructing and in-

terpreting graphs, and only a portion of the material covered in this topic will deal with per cent, an opportunity is provided to illustrate the importance of per cent in a wide variety of situations, such as:

Per cent of examples computed correctly and incorrectly.

Per cent of boys or girls in the class, in a grade, in a school, present, absent, and so on.

Reading and interpreting social studies data, such as per cent of various products produced in various regions, or countries, and so on.

Topic III. Using ratio in problem-solving

It has been noted that the three so-called "Cases" are not unique to per cent. Each one is a special application of a process common to decimal fractions and common fractions:

Case I. Finding a given part of a number.

Case II. Finding the ratio of one number to another.

Case III. Finding a number when part of it is given.

The purpose of this topic is to give the pupil special practice in each of these kinds of problems with common and decimal fractions which he has already learned to use, and then to extend the ideas to per cent, where the convenience of per cent in clarifying relationships becomes evident.

Topic IV. The parts of a percentage problem

Any percentage statement contains three elements: the number being compared, the number to which it is compared, and their ratio, expressed in per cent. In a percentage problem, one of the three is missing. Which is missing determines which of the three "Cases" the problem represents. A clear understanding of the relationships among the parts is needed if the pupil is to handle effectively the variety of problems involving per cent.

The percentage statement expresses this comparison:

15	is	30%	of	50.
dividend		quotient		divisor

If one of the three parts is missing, it can be found by using the relationships among the parts of a division operation:

Quotient = dividend ÷ divisor.

Divident = divisor × quotient.

Divisor = dividend ÷ quotient.

As a matter of convenience, the standard names for the three parts of a percentage statement should be used:

The *base* is the number to which another is compared (the divisor).

The *percentage* is the number being compared (the dividend).

The *rate* is the ratio expressed in per cent (the quotient when expressed as a decimal).

To highlight the elements in any problem, the pupil should write the percentage statement before undertaking to solve it. For example, the statement for three problems might be the following:

$$\underset{p}{9¢} \text{ is } \underline{\quad} \text{ per cent of } \underset{b}{30¢}.$$

$$\underset{r}{3\%} \text{ of } \underset{b}{\$3.50} \text{ is } \$\underline{\quad}.$$

$$\underset{p}{36¢} \text{ is } \underset{r}{10\%} \text{ of } \$\underline{\quad}.$$

In utilizing the relationships in problem-solving, the teacher may continue to use analysis, as in common and decimal fractions, or may introduce the percentage formula: $p = br$.

Until the formula has been mastered, the problems have been selected from familiar settings to emphasize the wide, everyday uses of per cent. Complexity and unfamiliarity of setting would present an added difficulty as the pupil is developing the basic concepts of per cent.

Placement of the topics

Topics I and II are commonly taken up in Grade 7. Newer textbooks are incorporating an introductory topic similar to Topic I, but not so comprehensive, in Grade 6. While some problems using per

cent are included in Grade 7, the treatment is intuitive and not extensive. Topics III and IV are allocated to Grade 8.

In General Mathematics, all four topics are distributed through the year. Experience has shown that even though the pupil has learned to use per cent in Grades 7 and 8, it can profitably be reintroduced in a study of ratio in Grade 9, with new applications. Between one topic and the next the bulletin board is used, and pupils are encouraged to bring in examples of the use of per cent for occasional discussion. In this way the study of per cent is directed toward development of language usage and sensitivity to applications as well as toward skill in computation.

From the axiomatic point of view, mathematics appears thus as a storehouse of abstract forms—the mathematical structures; and it so happens—without our knowing why—that certain aspects of empirical reality fit themselves into these forms, as if through a kind of preadaptation. Of course, it cannot be denied that most of these forms had originally a very definite intuitive content; but, it is exactly by deliberately throwing out this content, that it has been possible to give these forms all the power which they were capable of displaying and to prepare them for new interpretations and for the development of their full power.

It is only in this sense of the word "form" that one can call the axiomatic method a "formalism." The unity which it gives to mathematics is not the armor of formal logic, the unity of a lifeless skeleton; it is the nutritive fluid of an organism at the height of its development, the supple and fertile research instrument to which all the great mathematical thinkers since Gauss have contributed, all those who, in the words of Lejeune-Dirichlet, have always labored to "substitute ideas for calculations."—*N. Bourbaki, Translated by A. Dresden,* American Mathematical Monthly, *Vol. 68 (1950).*

A column of unofficial comment

What does '=' mean?

by H. Van Engen, Iowa State Teachers College, Cedar Falls, Iowa

Current literature contains such statements as "$a=b$ means that 'a' and 'b' are two names for the same number" (confining ourselves to the realm of numbers for the purposes of this discussion). There seems to be an element of ambiguity in such statements which should not appear in efforts to clarify the gibberish so often found in elementary mathematics texts. The careful reader will note that when we assert that "$a=b$ means 'a' and 'b' are names for the same number" we are using the symbols 'a' and 'b' as names, and, on the other hand, we are using them as variables. Now a variable cannot be a name, and a name cannot be a variable. Furthermore, there is a psychological difficulty for the unsophisticated student. The symbols 'a' and 'b' are not recognized as symbols for numbers, i.e., they are not standard numerals. They are not in the same category as the symbols '2,' '$\sqrt{3}$,' 'π,' etc.

Given a universe of discourse, U, consisting of the names for numbers, then the combination of symbols '$a=b$' must be a meaningless pattern until names, selected from U, are used to replace 'a' and 'b.' After making such a replacement, we made a decision as to whether the resulting statement is true or false. We can make this decision only if we recognize that the symbols which replace 'a' and 'b' are symbols for numbers.

Suppose the universe of discourse, U, is all proper fractions. Now we can use $a=b$ as a pattern to form statements about members of U. We obtain both true and false statements. Here are a few:

True Statements	*False Statements*
$1/2=2/4$	$2/3=1/2$
$5/6=10/12$	$1/2=1/3$

So, if we assert that $2/3=4/6$, we assert that these are two names for the same number. On the other hand, if we assert that $a=b$ we are only setting up a pattern —a kind of machine that can be used to make statements about numbers (the elements of our restricted universe of discourse).

Looking at algebraic expressions in this way leads one to appreciate the statement that "mathematics is the study of patterns." It is interesting to contemplate what effect, if any, this point of view should have on the teaching of first-year algebra.

Reviews and evaluations

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BOOKS

Mathematics for Science and Engineering, Philip L. Alger (New York: McGraw-Hill Book Company, Inc., 1957). Cloth, xi+360 pp., $6.95 (trade edition), $5.50 (text edition).

This text is a complete revision of *Engineering Mathematics* by the famous Charles P. Steinmetz. With the stated purpose of helping students and engineers use mathematics effectively, the book considers arithmetic, trigonometry, algebra, calculus, probability, etc., in successive chapters. It contains most of the mathematical procedures and methods that engineers and technicians need.

The reviewer was favorably impressed by the choice of material covered, the excellent figures, the adequate problem lists, and the bibliography at the end of each chapter. This is a useful synopsis of the mathematical courses of an engineer and should be a welcome addition to his library. It was disturbing to note the paradoxical statement on page 9, " . . . we cannot divide by zero, since this gives an indefinitely large, or infinite, result."—*C. H. Lindahl, Iowa State College, Ames, Iowa.*

Plane Geometry (revised edition), K. Bassler Keppler (New York: Frederick Ungar Publishing Company, 1957). Cloth, 359 pp., $4.75.

That this text looks different is one of the first conclusions even the most casual reader would reach. There are no pictures of planes cruising through the wild blue yonder, of pretty girls operating high-powered computing machines, nor of bridges spanning wide expanses of water.

Even more striking is the type of page and print. The book is large—359 pages—with each measuring about $7\frac{1}{2} \times 10\frac{1}{4}$ inches. The reproduction is apparently multilith, with black being the only color of print. The contents are subdivided into "books" rather than "chapters," with an introduction preceding Book One and a set of "Miscellaneous Exercises" and an "Appendix" succeeding the last book, Book Five.

None of this is by way of criticism, however. The reviewer's opinion is that this text is neither distinctly better nor radically worse than other texts. To expose the facts that led to this judgment, an analysis of the text follows.

The "Introduction" consists of the usual piling up and refining of facts, terminology, and symbolism. Of particular interest is an attempt to distinguish between physical and geometric entities, i.e., between material "substance" and "mental conception"; but it would seem that the student would be hard-put to preserve clearly this distinction, since the text fails to do so itself. (For example, on page 47 it is stated that "In general, the base of a triangle is the side on which it rests.") Also of interest is the preparation made for the deductive sequence of Book One via explanation and exercises dealing with the merits and defects of intuition and experiment, and the characteristics of definitions, assumptions, and theorems. But it seems that this introduction to deduction goes by too quickly with an insufficient number of examples to build up and clarify the important concepts involved. Also, some would find errors in this development, e.g., it is said that an assumption "must be stated in such a way that . . . its truth is evident"; a needless distinction is made between axioms and postulates; the importance of undefined words receives little emphasis. (See pages 33 ff., and the circularity in the straight line postulate and the definition of *distance* on pages 35 and 81, respectively.)

Book One—"Rectilinear Figures"—starts with the theorem concerning vertical angles being "equal," and ends at four inequality theorems dealing with triangles. En route, theorems concerning congruency, parallelism, locus, and quadrilaterals are covered. The special characteristics of this sequence seem to be these. Congruent triangle propositions are treated as theorems, with the techniques of rigid motion and superposition used. (Some, including the reviewer, prefer to postulate these in a high school geometry.) Throughout this and other books, an interesting use of "experiments" involving ruler and protractor is made to prepare the student for the plausibility of certain theorems. A distinction is made between this technique and that of construction with straight edge and compass, with the latter treated extensively (but in a deductive way only) rather than used for experiment as is done in some other texts.

The proposition that "if two parallel lines are cut by a transversal, the alternate interior angles are equal" is proved by establishing the congruency of two right triangles. This common technique is available since the parallel postulate and the method of *reductio ad absurdum* have been used to establish "if a line is parallel to one of two parallel lines, it is parallel to the other also," upon which depends the proposition

that "if a line is perpendicular to one of two parallel lines, it is perpendicular to the other also."

Noteworthy of comment in the rest of the text are such matters as the extensive concern with construction and with locus; a seemingly more extended than usual treatment of the commensurable and incommensurable cases (although proof of the incommensurable case for each theorem is deferred as requiring the calculus); an unusual though unsuccessful attempt to grapple with the important problem of order (between-ness, regions, etc.); a surprising coverage of constant, variable, and limit in order to obtain proofs for the circumference and area of a circle.

Some summary comments will imply the criteria used to arrive at the lukewarm opinion given earlier. Any student who successfully studied this text from cover to cover would be easily as well prepared as from a study of any other modern high school text. Indeed, he would probably have the advantage of having studied more than the usual amount of geometry. It is unfortunate that he would have had to plow his way through the usual semantic difficulties, would not have gained a clearer notion of the problem of between-ness, and would have found himself concentrating more on coverage than deliberating over important ideas. Greater and clearer emphasis upon the nature of a deductive sequence—i.e., upon direct and indirect proof, converse, the importance of the parallel postulate, the effect of a change in a postulate, a short deductive system—an earlier acquaintance with the pythagorean theorem, and some exposure to coordinate geometry would have helped this student. Also, he may have found the style of writing somewhat formal, with little humor, and no built-in motivation via interest-arousing problems.—*Robert Kalin, Florida State University, Tallahassee, Florida.*

Pure Mathematics, A First Course, J. K. Backhouse and S. P. T. Houldsworth (New York: Longmans, Green and Company, 1957). Cloth, xi+472 pp., $2.60.

This book was written and printed in England. It is intended to provide "a year's course in pure mathematics for those who have completed the usual syllabuses in elementary mathematics." In terms of American education, the content and level of presentation make this text suitable for high school seniors or for college freshmen.

The book consists of an introductory chapter on coordinates and the straight line, followed, in order, by six chapters on calculus, four on algebra, five on trigonometry, and three on coordinate geometry. An appendix gives the answers to the Questions which are interspersed throughout the text and to all problems in the customary Exercises, as in American texts. No tables are included.

The style is concise and lucid. However, American readers may find English usage somewhat confusing. For example, *index* is used instead of *exponent; gradient* is used instead of *slope* and also, initially, instead of *derivative*, and $\sqrt{2} = 1 \cdot 414$ is used instead of $\sqrt{2} = 1.414$.—*H. S. Kaltenborn, Memphis State University, Memphis, Tennessee.*

Putting Arithmetic to Work (Grade 7), Dale Carpenter and Elizabeth Cuthbertson (New York: The Macmillan Company, 1957). Cloth, 346 pp., $2.52.

Applying Arithmetic (Grade 8), Dale Carpenter and George F. Drake, Jr. (New York: The Macmillan Company, 1957). Cloth, 378 pp., $2.52.

The junior high school mathematics program is in a period of turmoil. The influence of the work in elementary mathematics in terms of the teaching of meaning and understanding and the development of modern approaches to high school mathematics have focused attention at the junior high level. This focus brings up the question: Should we as junior high mathematics teachers be looking for a text which will encourage the student's development of understandings in arithmetic and serve as an introduction to higher mathematics, or for a text which gives the student a "path to follow" to arrive at the correct answer for a series of textbook problems? This question needs to be answered before we, as teachers, can choose a text for our particular program. However, no matter what type of mathematics program we desire, there are certain principles which should guide our selection of a textbook. Among these principles are:

1. The text should show a consistent philosophy or "frame of reference."
2. The material in the text should be sound mathematically.
3. The content of the text should be consistent with material taught before and after its use.
4. The text should have an appeal to the user—both teacher and pupil.
5. A new textbook should reflect some progress in the teaching of mathematics.

The books under consideration are consistent in their development. The authors take the point of view that the following steps are necessary for mastery of the material: 1) The material is introduced with an example, 2) The example is worked for the student, 3) A series of questions is asked about the example, 4) A generalization is given in the form of a rule which the student can use in the solution of further examples, 5) Examples are provided to "assure the understanding of the generalization," 6) Maintenance practice is given later in the book to assure further mastery of the material. These six steps are used consistently throughout both the 7th- and 8th-grade textbooks.

The material in the books is sound mathematically, in that if the student follows the techniques in the text he will arrive at the cor-

rect answers. Some of the definitions and rules are vague. For example, in the unit in the 8th-grade text called "Using Algebra" it is said, "When we use letters to represent numbers in examples and problems we are using algebra." This is presented as a definition of algebra. Later in the same unit the equation is introduced by a scale balance where $3+2+1=4+2$, and this is called an equation. The definition given for an equation is, "When two quantities or expressions are equal, they form an equation." To introduce the solution of equations the statement $16=16$ is given and then it is noted that $8=8$, so we have the rule "When both sides of an equation are divided by the same number, the two quotients are equal." This type of development is typical of the algebra unit in the text. If the pupil follows the rules in the book he will arrive at the proper mathematical answers. If knowing the rules is the goal of mathematics, it can be achieved in this way.

The material in the two textbooks is organized to follow in a logical sequence. Enrichment materials and exercises are provided.

Throughout the entire series the teacher can determine the year for which the text is designed by counting the number of letters in the first word of the title. For example, the second-grade book is titled *We Work with Numbers* and the seventh-grade book is *Putting Arithmetic to Work*. Each book provides a thorough review of the preceding year and introduces the next year's work.

These texts should appeal to both teacher and students. From the teacher's standpoint the books are well constructed and well illustrated. The text is so organized that the beginning teacher can teach the material in order, with review provided in the book. There are tests in the book at the end of each unit and additional semester tests in the manual so that the teacher will not have to prepare tests.

The book should appeal to the student, for it will "catch the eye." The problems are designed with the maturity of the student in mind and examples are worked in the book so that the student has a "path to follow." The trick problems, which make up part of the enrichment, should be interesting to the better students.

These two textbooks were worth publishing and show progress in several ways. The material is well organized so that each part fits into the whole arithmetic program. Color is used throughout the book to call attention to important rules and examples. The problems are modern and should appeal to the junior high school student.

In summation, while the textbooks are an addition to materials available to the junior high teacher, they do *not* reflect a philosophy consistent with what we know about how pupils learn. For if it is the goal of the teaching of mathematics to develop generalizations, then the pupils should be given the freedom to formulate the generalizations for themselves. This freedom is severely limited when the generalizations are presented in a box in large type. Neither is understanding encouraged by the development of a step-by-step path which the student may follow to arrive at a "correct" answer.—*George Immerzeel, Iowa State Teachers College, Cedar Falls, Iowa.*

Many of the assignments for the slow learner seem to be "simple drill problems." They ranked second for the slow learner and above the average for the rapid learner. The teachers used the procedure, "Assign simple drill problems" (item 2, table 10), with the slow learner in the junior high school more than in the senior high school, but in the senior high school the method ranked third in extent of use. This survey indicates that, in the opinion of the heads of the departments of mathematics, drill ranks as one of the top differences. Perhaps this emphasis on drill might raise questions concerning its importance and place in mathematics education. Will drill result in understanding or should understanding be the basis for drill? Will drill without understanding develop undesirable attitudes and prevent effective learning? Do understandings as well as manual skills require repetition in order to be retained?—*Teaching Rapid and Slow Learners in High Schools the Status of Adaptations in Junior, Senior, and Regular High Schools Enrolling More than 300 Pupils.*—Bulletin 1954, *No. 5, Office of Education, U. S. Department of Health, Education, and Welfare.*

• TESTING TIME

Edited by Robert S. Fouch, Florida State University, and Robert Kalin, Florida State University, Tallahassee, Florida

A test for senior day

by Wallace Manheimer, Franklin K. Lane High School, Brooklyn, New York

The following test was given to my advanced algebra class on Senior Day when high spirits and fancy costumes were much in evidence. After a preliminary groan or two, the class awoke to the fact that the test was something of a mathematical joke, and they entered gaily into the spirit of it.

The test is given below, along with solutions.

ADVANCED ALGEBRA
TEST IN HONOR OF SENIOR DAY

Note: There will be a limit of one square inch of scrap paper to each student.

1. Explain why $x^5-7x^4+3x^3-9x^2+2x-17$ does not have four imaginary roots.
2. Write in the form $\cos w+i \sin w$ one of the imaginary 360th roots of 1.
3. Evaluate

$$i^{3765}+i^{3764}+i^{3763}+i^{3762}.$$

4. Express

$$\frac{3i-3}{1-i}$$

in the form $a+bi$.

5. Write the number of imaginary roots of $x^6+3x^4+2x^2+x=0$.
6. Write the number of complex roots of

$$x^{23}-17x^{18}+27x^{15}-14x^{11}+26x^{10}-x^9+37x^4-2x^3+3x^2-x+9=0.$$

7. One root of

$$12x^4-20x^3-309x^2-20x+12=0$$

is 6. What is another?

8. Find a root of

$$7x^7-.1x^6-.01x^5-.001x^4-.0001x^3-.00001x^2-.000001x-.0000001=0.$$

9. Solve

$$x^5+5x^4+10x^3+10x^2+5x+1=0,$$

given that it has five rational roots.

10. Write the equation whose roots exceed by 1 those of the equation in Example 9.

Solutions

1. Of course the expression has no roots since it is not an equation.
2. If w is made an integer it is impossible to give a wrong answer.
3. Since the four exponents are consecutive integers the expression gives the four powers of i, whose sum is zero.
4. $3i-3=-3(1-i)$. Hence, the answer is -3. (Many students multi-

plied by the conjugate of the denominator.)

5. Four. The equation has two real roots, one of which is zero. Descartes' rule of signs had better be applied cautiously.
6. Twenty-three. All the numbers studied, including real numbers and pure imaginaries, are elements of the set of complex numbers.
7. 1/6 is another root, since the equation is identical to the one having reciprocal roots.
8. .1 is a root, since 1 obviously is a root of the equation that results when the roots are multiplied by 10.
9. The roots are all rational and factors of 1. The equation has no positive roots; hence all are equal to -1. Some students observed that the left-hand side is the binomial expansion of $(x+1)^5$.
10. The roots will all equal 0; hence, the equation is $x^5=0$. Students who used the usual algorithm for increasing the roots were somewhat crestfallen at hearing this method.

Results were so good that another test of the same type is planned for the next Senior Day. There is a bit of fun in mathematics—if only we will look for it.

Letter to the editor

Editorial Note: This letter calls our attention to some of the problems teachers face if they want to go back to school to learn the "new mathematics." A copy of this letter was sent to the editor of THE MATHEMATICS TEACHER.

Dear Dr. Fehr,

I am a mathematics teacher, a member of the National Council of Teachers of Mathematics, and seriously interested in keeping informed of 20th century mathematics in order to do a better teaching job. Since I have been fortunate enough to work with gifted high school seniors, I find that I should familiarize myself with such current topics as programming digital computers and elementary set theory. In an effort to obtain this training, while at the same time working toward a graduate degree, I made the rounds of the local universities and found that the following condition exists:

1. A mathematics teacher can work either for an M.A. in mathematics or an M.A. in education.
   a. If the program toward an M.A. in math is chosen, the candidate is expected to take very advanced work, actually intended for engineers or mathematicians. The undergraduate program of a high school math teacher does not prepare him for that type of graduate program.
   b. If the program toward an M.A. in education is chosen, the candidate is advised to take courses in education in preparation for the required comprehensive examination.
2. There exists no reciprocal agreement between different universities.
   a. To give a specific example: The George Washington University does not offer a course in programming. The Graduate School of the Department of Agriculture, however, does offer a course in programming. If a candidate at the George Washington University takes a course at any other institution, the work cannot be counted toward the required 30 semester hours of residence at G. W. I was informed that if I were to take 15 semester hours at another institution, the credits would be "counted," but the requirement raised to 45 credits.
3. Some universities (Chicago, Rutgers, Syracuse, and others) offer excellent programs for math teachers. But how many teachers can afford the time and money, knowing that the credits earned cannot be applied toward the residence requirements of the institution at which they matriculated?

It seems to me that unless the NCTM makes its leadership felt and campaigns for more realistic programs of teacher training, the shortage of qualified math teachers will continue and grow worse. I sincerely believe that you, as president of the NCTM, could and should awaken the university administrators to this unfair and antiquated system.

Sincerely yours,

Mrs. John B. Wolff
8712 Plymouth Street
Silver Spring, Maryland

• NOTES FROM THE WASHINGTON OFFICE

Edited by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

Membership statistics

An important responsibility of the Washington Office is the keeping of complete and accurate statistics. Recently we compiled some up-to-date data on our membership and subscription lists. The following figures may be of interest to members of the Council.

Distribution of subscriptions

Our last check of the distribution of subscriptions to our journals among our members showed the following facts:

87% take The Mathematics Teacher.
35% take The Arithmetic Teacher.
22% take both journals.

A similar check made two years ago showed corresponding per cents of 90%, 30%, and 20%. As the membership list has grown, both journals have grown in circulation, but these figures show that The Arithmetic Teacher is growing more rapidly.

Distribution of expirations

Many organizations define a membership or subscription year, so that all expirations take place at the same time. It has become a tradition in the Council, however, to permit each member as he joins to state the month or issue with which he desires to have his membership begin. This means that expirations take place throughout the year. During recent years the memberships and subscriptions have been expiring according to the following pattern:

50% expire at the end of the school year.
30% expire during the fall months, October through December.
20% expire during the spring months, January through April.
10% are paid in advance before expiration and thus do not receive the usual expiration notice.

Per cent of loss per year

It is rather difficult to determine, without setting up laborious procedures, the number of memberships that die or are not renewed at all during a year. Unrenewed plates are discarded because of lack of storage space after they become a year old. At the end of the 1955–56 school year the number of plates discarded was 25% of the number of members and subscribers during that year. However, not all of these plates represented discontinued memberships. Some of the discarded plates had been replaced by new plates at new addresses.

We are sure that our mortality, or per cent of loss, during a year is between 20% and 25%. In other words, one member out of four or five drops out of the organization each year. Offhand this seems like a high mortality. However, the mortality in the entire teaching profession is about 8% per year. That is, 8 out of 100 teachers leave the profession each year because of death, retirement, change of occupation, or other causes. Thus, a yearly loss of 8% is the *minimum* loss that any organization of teachers could experience.

The experience of other professional subject-matter organizations is very much like ours. Even the general professional organization for teachers, the National Education Association, experiences a loss of one out of five or six each year.

Although our per cent of loss is not unusual, it would certainly be advantageous, both to the Council and to the individual member, if a larger per cent of our members could be induced to remain in the organization each year. This is a problem in which every reader of The Mathematics Teacher can be of assistance.

Report of the Nominating Committee

The Committee on Nominations and Elections presents its slate of nominees for offices to be filled in the 1958 election. The term of office for the president and the two vice-presidents is two years. Three directors are to be elected for terms of three years.

In making nominations for the three director positions, the Committee followed the directive adopted by the Board of Directors in 1955 which states, "Nominations shall be made so that there shall be not more than one director elected from each state, and that there shall be one director, and not more than two, elected from each region." Members may consult THE MATHEMATICS TEACHER for October, 1955, for a map of the regions as they are now defined.

Since the Southwestern Region will have no carry-over director, it was necessary for the Committee to nominate two persons from that region to avoid making nomination equivalent to election.

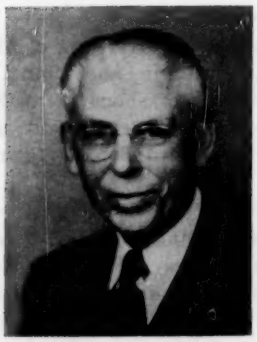
The following rule, adopted by the Committee, to determine who shall be declared elected as directors, will be followed: The three nominees receiving the largest number of votes will be declared elected, provided that at least one of them is a nominee from the Southwestern Region; otherwise the two nominees receiving the largest number of votes will be declared elected, and of the two nominees from the Southwestern Region the one who receives the larger number of votes will be declared elected.

Ballots will be mailed on or before February 19, 1958 from the Washington Office to members of record as of that date. Ballots returned and postmarked not later than March 19, 1958 will be counted.

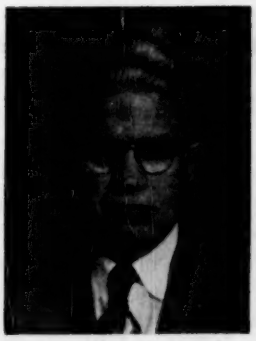
The Committee wishes to thank the many members of NCTM for their help in giving their suggestions for nominees. It is hoped that all members of our organization will exercise their privilege of voting.

CLIFFORD BELL, *Chairman*
JACKSON ADKINS
MARY FOSTER
WILLIAM GAGER
ROBERT PINGRY
IRENE SAUBLE
VERYL SCHULT
MARIE WILCOX
LYNWOOD WREN

NOMINEES FOR PRESIDENT



H. GLENN AYRE



HAROLD P. FAWCETT

H. Glenn Ayre

Professor of Mathematics, Head of Department of Mathematics, and Director of General College Division, Western Illinois University, Macomb, Illinois.

Ed.B., Southern Illinois University; S.M., University of Michigan; Ph.D., Peabody College for Teachers.

Instructor in rural schools, Marion County, Illinois; High School, Carterville, Illinois; High School, Benton, Illinois; High School, Waukegan, Illinois; Supervisor of Student Teaching in Mathematics, Professor of Mathematics, and Head of Department, Western Illinois University, Macomb, Illinois.

Member: NCTM and CASMT since 1924; AAAS; NEA; Illinois Education Association; MAA; Phi Delta Kappa; and other educational societies. Listed in *American Men of Science, Who's Who in Education, Who's Who in the Midwest.* Fellow of the American Association for the Advancement of Science.

Activities in NCTM: Vice-President, 1953–55; Chairman, Committee on Affiliated Groups, 1954–56; Program Chairman, Summer Meeting in Seattle, 1954; Editor, Handbook Committee, 1954; Editor, Affiliated Groups Newsletter, 1954; Member, Executive Committee, 1954; Chairman, Committee on Free and Inexpensive Materials, 1953; Member, Algebra Committee for *Seventeenth Yearbook;* Secretary of Symposium on Teacher Education in Mathematics, University of Wisconsin, 1952; Publicity Committee, 1952; speaker and discussion leader for many NCTM programs.

Other Activities: President, Illinois Council of Teachers of Mathematics, 1955–56; Member of Executive Council, 1948–53; Vice-President and Chairman, Illinois Section MAA, 1951–53; Member, Joint Committee of Illinois Section MAA and NCTM to Study Teacher Training in Mathematics; Member, Committee on Tests for ICTM and Illinois Curriculum Project; Chairman, Mathematics Section and Junior College Section CASMT, 1941, 1947; speaker and discussion leader for numerous institutes and workshops in mathematics education.

Publications: "An Analysis of Individual Differences in Plane Geometry," "An Analysis of the Performance of College Freshmen on Arithmetic," *Basic Mathematical Analysis;* numerous articles in professional journals; Coauthor of *A First Course in Coordinate Geometry,* 1956.

Harold P. Fawcett

Professor of Education, Ohio State University, Columbus, Ohio.

A.B. (honors in mathematics), Mt. Allison University, Sackville, New Brunswick, 1914; graduate study, University of California, 1915–17; A.M., Columbia, 1924; Ph.D., Columbia, 1937.

Junior and Senior High School, Fort Fairfield, Me., 1914–15; YMCA Evening School, San Francisco, 1915–17; Field Artillery, France, 1917–19; Home Study Division, YMCA Schools, New York, 1919–24; Instructor, Columbia University, 1924–32; Assistant Professor, Ohio State University, 1932–37; Associate Professor, Ohio State University, 1937–43; Professor, Ohio State University, 1943—; Associate Director, University School, Ohio State University, 1938–41; Chairman, Department of Education, Ohio State University, 1948–56; Visiting Professor, Northwestern University, Columbia University, Universities of Michigan, Virginia, Wisconsin, and Utah; Director of numerous workshops in mathematics education, lecturer, consultant, and educational adviser to Coronet Instructional Films.

Member: NCTM; Ohio Council of Teachers of Mathematics; Ohio Education Association; National Education Association; American Association of University Professors; National Society for the Study of Education; The John Dewey Society; Canadian Universities Associa-

tion; The Torch Club; Phi Delta Kappa; Kappa Phi Kappa.

Activities: Former director of NCTM; Ohio State Mathematics Committee; 1934–35; Committee on Secondary School Curriculum, 1936–38; Committee on Experimental Units, 1942——; Commission on Research and Service, North Central Association, 1943——; Committee on Higher Education, Ohio College Association, 1957——.

Publications: "The Nature of Proof" (*Thirteenth Yearbook* of the NCTM), *Mathematics in General Education* (committee member), and contributions to such journals as THE MATHEMATICS TEACHER, *School Science and Mathematics, Ohio Schools, Educational Research Bulletin, North Central Association Quarterly, The English Journal, California Journal of Secondary Education*, and *The School Executive.*

NOMINEES FOR VICE-PRESIDENT—SENIOR HIGH SCHOOL LEVEL

IDA MAY BERNHARD

RACHEL P. KENISTON

Ida May Bernhard

Consultant in Secondary Education, Texas Education Agency (State Department of Education), Austin, Texas.

A.B., M.A., University of Texas; summer sessions at University of Vermont and Teachers College, Columbia University.

Teacher in Texas Public Schools, 1927–45; Supervisor of Mathematics in Laboratory School, Southwest Texas State Teachers College and San Marcos High School, San Marcos, Texas, 1945–52; attended Duke University Mathematics Institute, 1946–52; Study Group Leader Duke University Mathematics Institute 1950–52; 1954–55; University of California, Los Angeles, California Conference for Teachers of Mathematics, 1951–57; University of Houston Mathematics Institute, 1952–54; University of Texas Mathematics Conference, 1953; Louisiana State University Mathematics Institute, 1950, 1952–55; Consultant, Texas State College for Women, Mathematics Workshop, 1953–56.

Member: NCTM; NEA; AAAS; NASCD; MAA; Texas State Teachers

Association (Vice-President, Alamo District II, 1951–52; President, Alamo District II, 1952–53); State Honorary Member of Delta Kappa Gamma; Texas State Textbook Committee, 1951; Texas Association of Supervision and Curriculum Development; Texas Association of Elementary Principals; Texas Council of Teachers of Mathematics.

Activities in the NCTM: member of Board of Directors, 1952–55, 1955–58; Southwestern Area Representative for Affiliated Groups Committee, 1952–55; appearance on convention programs; participation on committees.

Publications: "Materials Available for Counseling in Mathematics," THE MATHEMATICS TEACHER, April, 1954. Served as leader of committee which prepared Texas Education Agency Bulletin 548, *Suggestions for Teachers of Mathematics*, 1953.

Rachel P. Keniston

Teacher at Stagg High School, Stockton, California.

A.B., Smith College, Northampton, Mass.; M.A., College of Pacific, Stockton, California.

Member: NCTM; California Mathematics Council; New England Association for Teachers of Mathematics; NEA; California Teachers Association; Delta Kappa Gamma.

Activities: Former vice-president, California Mathematics Council; currently county representative of California Mathematics Council; member NCTM Committee of Supplementary Publications; speaker at numerous mathematics meetings both state and NCTM; group leader (instructor) for a week's course at summer meeting of NEATM in 1955; locally, Chairman of Mathematics Committee for Stockton, California; workshops and summer sessions, College of Pacific, Columbia, Plymouth (New Hampshire) State Teachers College, University of California, Los Angeles.

Publications: Coauthor of textbook: *Plane Geometry;* article in *Eighteenth Yearbook* of the NCTM; articles in *Bulletin, California Mathematics Council.*

NOMINEES FOR VICE-PRESIDENT—ELEMENTARY SCHOOL LEVEL

MARGUERITE BRYDEGAARD

E. GLENADINE GIBB

Marguerite Brydegaard

Associate Professor of Education, San Diego State College.

A.B., San Diego State College; M.A. (and advanced graduate study), The Claremont Graduate School, Claremont, California; travel in Europe to study art and to investigate methods for teaching mathematics in the elementary school.

Associate Editor, *The Arithmetic Teacher*, 1956——; Editorial Board of the Association for Childhood Education International, 1954–56; leader of elementary section, California Conference for Teachers of Mathematics, University of California, Los Angeles; visiting instructor, The Claremont Graduate School; Director of Summer Conferences on Teaching of Mathematics, San Diego State College; led many workshops and presented reports and studies to local, state, and national groups of educators; presented papers at several National Council of Teachers of Mathematics meetings; presented a paper at the University of Wisconsin Mathematics Conference, 1953; worked on elementary-level testing at the Educational Testing Service Workshop, 1955.

Member: National Council of Teachers of Mathematics, Kappa Delta Pi; Pi Lambda Theta; California Mathematics Council; Life Member Association for Childhood Education International.

Activities in NCTM: Member, National Council of Teachers of Mathematics Committee on Supplementary Publications, 1955—, Place of Meeting Committee, 1956——.

Publications: articles include "Creative Teaching Points the Way," first issue, *Arithmetic Teacher;* article in *The Bulletin* (National Association of Secondary School Principals), 1954; coauthor (with Howard Fehr, Leonore John, Ann Peters) *Step Test —Elementary Level;* Coauthor of *Building Mathematical Concepts in the Elementary School*, 1952.

E. Glenadine Gibb

Associate Professor of Mathematics, Iowa State Teachers College, Cedar Falls, Iowa.

B.Ed., Western Illinois University, Macomb, Illinois; M.A., George Peabody College for Teachers, Nashville, Tennessee; Ph.D., University of Wisconsin, Madison, Wisconsin.

High School mathematics teacher, Mendon, Illinois (1941–1945), Geneseo, Illinois (1945–1946); Instructor in mathematics to associate professor of mathematics, Iowa State Teachers College, (1946——); Visiting lecturer, University of Wisconsin (Summer, 1951), University of Vermont (Summer, 1955), Marshall College, Huntington, West Virginia (Summer, 1957); Director of Arithmetic Workshop, Marshall College, 1950, 1951, 1957; Member of institute staffs—Rutgers University (1954), Sedgwick County, Kansas (1954), Louisiana State University (1955).

Member: NCTM; Past member of Board of Directors, Chairman of Policy Committee, Chairman of Elementary Mathematics Section, Central Association of Science and Mathematics Teachers; past vice-president and past president, Iowa Association of Mathematics Teachers; American Educational Research Association; National Education Association; Iowa State Education Association; American Statistical Association; American Association of University Professors; American Association of University Women; Delta Kappa Pi; Pi Lambda Theta; Kappa Delta Pi.

Activities in NCTM: Iowa State Representative; Associate Editor of *Arithmetic Teacher;* speaker and discussion leader on many convention programs.

Publications: Coauthor of *General Mental Functions Associated with Division and Teaching Children to Divide;* articles in *Review of Educational Research, Journal of Experimental Education*, and *Arithmetic Teacher.*

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Chairman of Department of Mathematics, Lyons Township High School, La-Grange, Illinois.

B.Ed., Southern Illinois University; M.S., University of Iowa; additional graduate study at University of Illinois.

Teacher of Mathematics, Sparta Township High School, Sparta, Illinois; Urbana High School, Urbana, Illinois; Thornton Fractional Township High School, Calumet City, Illinois; Trial Judge Advocate and Claims Officer, Army of United States, 1942–46.

Member: NCTM; CASMT; NEA; MAA; Illinois Council of Teachers of Mathematics; American Association for the Advancement of Science; Men's Mathematics Club of Chicago. Listed in *Who's Who in American Education*.

Activities in NCTM: Committee member for mathematics number of *The Bulletin* (National Association of secondary School Principals), May, 1954; Chairman, Secondary School Curriculum Committee of the NCTM, 1955——; Associate Editor of *Mathematics Student Journal*, 1953–54.

Other Activities: President, Men's Mathematics Club of Chicago, 1952–53;

President, Illinois Council of Teachers of Mathematics, 1954–55; numerous speeches and papers at mathematics conferences and teachers' institutes.

Publications: "Mathematics Tomorrow," *NEA Journal*, May, 1957; "Building a Mathematics Program," THE MATHEMATICS TEACHER, April, 1956.

J. Houston Banks

Southeastern Region

Associate Professor of Mathematics, George Peabody College for Teachers.

B.S., Tennessee Polytechnic Institute; M.A. and Ph.D., George Peabody College for Teachers.

Elementary schoolteacher and principal; high school mathematics instructor; high school principal; junior college dean and mathematics instructor; Mathematics Instructor, 133rd Detachment, Army Air Corps; Professor of Mathematics, Florence State Teachers College, Alabama; Visiting Professor of Education, Alabama Polytechnic Institute.

Member: NCTM; MAA; NEA; Tennessee Teachers of Mathematics; Tennessee Academy of Science; TEA; Phi Delta Kappa; Kappa Delta Pi; Pi Mu Epsilon.

Activities: Member, Committee on Supplementary Publications; Southeastern Representative for Affiliated Groups; Speeches and papers at mathematics conferences, National Council, and other professional meetings; Consultant assignments with local school systems; Active in organizing Tennessee Teachers of Mathematics, President, 1954–55; Active part in organizing and conducting state-wide high school mathematics contests; Cochairman, committee on high school contests, Southeastern Section, MAA; Former President, Mathematics Section, TEA; Former State Representative, NCTM; Research in relative effectiveness of various types of mathematics programs.

Publications: Contributor to THE MATHEMATICS TEACHER, *Tennessee Academy of Science Journal;* Author, *Mathematics for General Education* and *Elements of Mathematics.*

Burton W. Jones

Southwestern Region

Professor of Mathematics and Chairman, Department of Mathematics, University of Colorado, Boulder, Colorado.

B.A., Grinnell College; M.A., Harvard University; Ph.D., University of Chicago.

Instructor in Mathematics, Western Reserve University, 1924–26; Instructor in Mathematics, University of Chicago, 1928–29; National Research Fellow, 1929–30; Assistant Professor, Associate Professor, and Professor of Mathematics, Cornell University, 1930–48; Professor of Mathematics, University of Colorado, 1948——; Chairman of the Department, 1949——.

Member: NCTM; MAA; AMS; AAUP; AAAS; Sigma Xi; and Phi Beta Kappa.

Activities in NCTM: Member, Secondary School Curriculum Committee.

Other Activities: Second Vice-President, MAA; Chairman, Committee on Visiting Lectureships, MAA; Regional Consultant in Science and Mathematics, STIP of AAAS.

Publications: *A Table of Einstein-Reduced Positive Ternary Quadratic Forms; Elementary Concepts of Mathematics; Arithmetic Theory of Quadratic Forms; Theory of Numbers;* numerous articles in mathematical journals.

Eunice Lewis

Southwestern Region

Supervisor of Mathematics, University of Oklahoma High School, and Assistant Professor of Education, University of Oklahoma.

A.B., M.A., University of Oklahoma.

Teacher of mathematics at Covington High School, Covington, Oklahoma; Sapulpa High School, Sapulpa, Oklahoma; Cleveland Junior High School and Central High School, Tulsa, Oklahoma.

Member: NCTM; NEA; Oklahoma Education Association; Oklahoma Council of Teachers of Mathematics; Kappa Delta Pi; Delta Kappa Gamma; MAA; Pi Mu Epsilon; Association for Student Teaching.

Activities in NCTM: On program at several meetings of the NCTM; State Representative; Southwestern Regional Representative; Past President of the Oklahoma Council of Teachers of Mathematics; member of committees.

Other Activities: Member of the staff of summer workshops at Louisiana State University, University of Oklahoma, Oklahoma State University, Central State College at Edmond, Oklahoma; Consultant, Oklahoma Study of the Mathematics Curriculum of the State; speaker at local, district, and state meetings of Mathematics Teachers.

Publications: In THE MATHEMATICS TEACHER, "An Experience Program for the Training of Teachers of Mathematics at the University of Oklahoma," "The Pupil Discovers Algebra," "The Role of Sensory Materials in Meaningful Learning." In the 1955 *Association for Student Teaching Yearbook*, "How I Gave My Student Teachers Experience in Establishing and Maintaining Desirable Teacher-Pupil Relations."

Bruce E. Meserve

Northeastern Region

Professor of Mathematics and Chairman, Department of Mathematics, New Jersey State Teachers College at Montclair.

A.B., Bates College; A.M., and Ph.D., Duke University.

Teacher of Mathematics, Moses Brown School, 1938–41; Graduate work at Duke University, 1941–42, 1945–46; Civilian Public Service, 1942–43; U. S. Army, 1943–45; Instructor of Mathematics, University of Illinois, 1946–48, Assistant Professor, 1948–54; Associate Professor of Mathematics, New Jersey State Teachers College at Montclair, 1954–57, Professor of Mathematics and Chairman of Mathematics Department, 1957——; Lecturer at Duke Institute, 1942, New England Institutes, 1952, 1953, 1955, 1956; National Science Foundation Institute at Oklahoma A. and M. College, 1955; New Jersey Institute, 1956, 1957; and at more than twenty other professional groups.

Member: Association of Mathematics Teachers of New Jersey—Executive Board, 1954——, Book Review Editor, 1956——, Committee on Contests; Advisory Committee for the New Jersey Mathematics Institute; Illinois Council of Teachers of Mathematics—Vice-President for Colleges, 1952–54, Chairman of Committee on the Strengthening of the Teaching of Mathematics, 1953–54; Association of Teachers of Mathematics in New England; Executive Committee of New Jersey Section of MAA; American Mathematical Society; Committee on Contests and Awards of Metropolitan New York Section of MAA, 1954–57; Phi Beta Kappa; Sigma Xi; and Kappa Mu Epsilon.

Activities in NCTM: Member of Committee on Research, 1953–54, Chairman, 1954–55; Committee on Institutes and Summer Workshops, 1954–55; Chairman of Committee on the Nomination of an Editor for THE MATHEMATICS TEACHER, 1955; Secretary of Committee on Coordination of Mathematics with Business and Industry, 1954–55, Chairman, 1955–1957; Member of Editorial Committee of *Twenty-Third Yearbook*, 1954–57; Member of Committee on Secondary School Curriculum, 1955——; Chairman, Subcommittee on Secondary School Standards, 1956——.

Publications: Author of *Fundamental Concepts of Algebra* and *Fundamental Concepts of Geometry;* Fourth edition of *College Algebra* (with P. M. Whitman); Co-author of the University of Illinois Bulletin entitled *Mathematical Needs of Prospective Students at the College of Engineering of the University of Illinois;* Author of over twenty articles (five jointly with

other authors) appearing in THE MATHEMATICS TEACHER and other professional journals.

Elizabeth J. Roudebush
Western Region

Director of Mathematics from Kindergarten through Grade Twelve for Seattle Public Schools, Seattle, Washington.

Graduate of State College of Washington, Pullman, Washington; A.M., Teachers College, Columbia University; summer sessions at University of Washington, University of California at Los Angeles, and University of Southern California.

Teacher, Mathematics, Roosevelt High School, Seattle, Washington; Mathematics Department Head, Edison Technical School, Seattle, Washington; Director of Mathematics, Seattle, 1949——. During World War II served in WAVES as Women's Reserve Representative at U. S. Naval Hospital in Chelsea, Mass.

Member: NCTM; ASCD; NEA; Washington Education Association (Past President of local group affiliated with WEA and NEA); Delta Kappa Gamma (Past Vice-President of local chapter); Pi Lambda Theta (Past President of local alumnae chapter); Soroptimist International (President, Seattle Club).

Activities in NCTM: Member, Board of Directors, 1953–56; Regional Representative of Western Region of Affiliated Groups; Chairman of Place of Meetings Committee; Cochairman in Charge of Local Arrangements for 1954 Summer Meeting; Section Speaker at 1953 Christmas Meeting and 1956 Annual Meeting; Chairman of Affiliated Groups, 1956–59.

Other Activities: helped organize Puget Sound Council of Teachers of Mathematics; helped organize Washington State Mathematics Council.

Publications: *Laboratory Geometry;* "An Arithmetic Bulletin for Parents," THE MATHEMATICS TEACHER, May, 1951; "Professional Classes of the Seattle Public Schools," *Twenty-Second Yearbook* of NCTM.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

ANNUAL MEETING

April 9–12, 1958
Hotel Cleveland, Cleveland, Ohio
Lawrence Hyman, Board of Education, Cleveland, Ohio

JOINT MEETING WITH NEA

June 30, 1958
Cleveland, Ohio
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

EIGHTEENTH SUMMER MEETING

August 19–20, 1958
Colorado State College of Education, Greeley, Colorado
Forest N. Fisch, Colorado State College of Education, Greeley, Colorado

CHRISTMAS MEETING

December 29–30, 1958
Sheraton-McAlpin Hotel, New York, New York
Abraham S. Kadish, High School of Fashion Industries, 225 West 24th Street, New York 11, New York

Other professional dates

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University of Chicago, Chicago 37, Illinois
E. P. Northrop, Eckhart Hall, Box 23, University of Chicago, Chicago 37, Illinois

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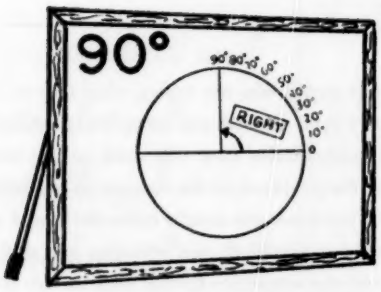

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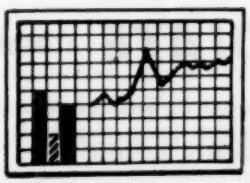

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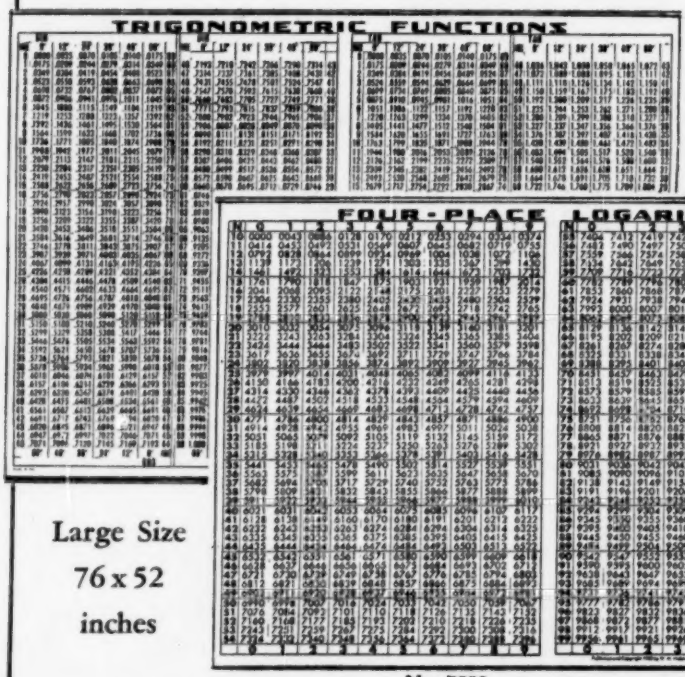

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